Incompleteness Shocks*

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Abstract

This paper studies the effects of shocks to the degree of market completeness. We present a dynamic stochastic economy where agents can trade in complete markets in normal times, but where financial markets can stochastically become incomplete. When this happens, agents cannot trade in state contingent assets and cannot re-hedge their risks. Our model formalizes a new type of purely financial shock, which we call an incompleteness shock. Even if we allow our agents to hedge the incompleteness shock itself, we find that these shocks are sufficient to trigger a recession with misallocation of capital, lower aggregate output, and consumption. Our results highlight that financial market disruptions will unavoidably generate a recession, even if they are perfectly anticipated and agents can freely reallocate resources ex-ante.

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1 Introduction

Most existing research on incomplete markets considers models in which the degree of incompleteness is constant over time. This is a natural setup when one is interested in understanding the consequences of market incompleteness for the propagation of shocks. For instance, there is a long literature in macroeconomics that explains how financial market frictions can act as a propagation and amplification mechanism for real and nominal shocks that originate outside the financial sector. A more recent body of work emphasizes the role of shocks that originate in the financial sector as direct sources of economic fluctuations. These shocks typically take the form of destruction of intermediary capital or changes in the tightness of borrowing constraints.

Compared to the existing literature, we are interested in understanding the consequences of shocks to the degree of market completeness. We think that this is a useful undertaking for two main reasons. A first reason is that we will be able to formalize a new type of shock, and trace its effects on the economy. This shock is arguably relevant because it captures the idea that hedging opportunities can disappear during a crisis.

Another reason that leads us to think that this line of research is useful is that it allows us to revisit the macro-hedging puzzle. It is commonly argued that amplification and propagation requires two set of frictions, both ex-ante and ex-post frictions. There must be frictions that prevent agents from arranging transfers ex-ante contingent on the realization of an aggregate shock, as well as frictions that prevent agents from freely transferring resources from and towards the future after the realization of an aggregate shock. Kiyotaki (2011) succinctly summarizes this common wisdom:

"Therefore, in order to justify (financial frictions as) the propagation mechanism, we need to first explain why firms would choose to borrow and not issue contingent securities in procuring their funds. We also need to explain why firms choose not to issue common stocks in order to recover net worth when it is deteriorating."

In this paper, we argue that shocks to the ability of financial markets to hedge future risks, even if anticipated and perfectly contractible, are sufficient to cause an economic downturn that features an inefficient allocation of capital and a reduction in aggregate output and consumption. We refer to these shocks as *incompleteness shocks*. Our results highlight that a disruption of financial market trading is sufficient to generate a recession even if all agents can perfectly arrange transfers of resources ex-ante. A broad interpretation of our results supports the idea that disruptions in financial markets will unavoidably generate recessions. The crucial assumption that makes our results non-trivial is the fact that incompleteness shocks are fully hedgeable ex-ante, in the sense that agents can arrange transfers ex-ante contingent on the realization of a financial shock.

We present our results in a parsimonious two-agent model of capital allocation and risk-sharing. There are two types of agents in our model, experts and households, who can trade capital every period in frictionless spot markets. Both experts and households have identical preferences, but they experience different shocks to their production technology. In a first-best scenario, experts and households trade capital and financial securities every period in a way that equalizes their marginal productivities of capital. The ability to hedge all risks in that case implies that capital is optimally allocated across agents at all times.

Instead, in our model, the ability to trade in financial markets varies over time. A Markov process determines the oscillations in the economy between normal periods and incompleteness periods. In normal periods, both agents can trade a complete set of one-period Arrow-Debreu securities; we say that markets are complete at those times. With some positive probability, the economy transitions into a new regime in which agents can only trade a non-contingent bond; we say that the economy experiences an incompleteness shock when that state materializes.[1] The economy returns to normal times with a certain probability.

We formally show that, if financial markets are complete at a given period, the allocation of capital in the economy is efficient in that period. More importantly, we show that, after an incompleteness shock hits the economy, capital will generically be misallocated among agents, reducing aggregate output. The degree of misallocation depends on the size of an incompleteness wedge, which captures the differences in the continuation value of capital for experts and households. We decompose the incompleteness wedge, which can be traced back directly to the set of available financial markets, into a risk-free and a risky component. We show that both components may generate an inefficient allocation of capital. Intuitively, the inability to borrow in risk-free debt and the inability to hedge some risks induce some agents to hold less capital than in the first-best allocation. We develop a pair of three-period examples to analytically illustrate how the risk-free wedge and the risky wedge operate in simple environments.

Crucially, we assume that in normal times experts are able to contract on the incompleteness shock ex-ante, so households can in principle transfer any given amount of resources to experts conditional on the realization of the incompleteness shock. Even though agents make use of this possibility, potentially dampening the effect of an incompleteness shock, we show this is in general not sufficient to eliminate the misallocation of capital and the reduction in output caused by an incompleteness shock. Intuitively, this occurs because the allocation of capital across agents is determined by forward-looking forces, so they existence of residual unhedgeable risk looking-forward is necessarily associated with a misallocation of capital.

We quantitatively solve the model to illustrate the effects of incompleteness shocks. In our quantitative results, we make experts natural holders of capital by assuming that experts are always more productive than households, but that their technology is risky. We assume that households are less productive than experts, but that their technology features a constant productivity. This formulation forces experts to bear more risk after an incompleteness shock than when financial markets are well-functioning. Even though they have access to a non-contingent bond, the inability to hedge induces experts to sell capital to households and to imperfectly self-insure after an incompleteness shock. Intuitively, since the incompleteness shock prevents experts from hedging future risks, they endogenously become more risk averse, selling capital to low valuation users. This relocation of capital

[1] We also consider the case in which all financial markets cease to operate after an incompleteness shock.

is inefficient and reduces aggregate output by reducing aggregate TFP.

Our current simulations indicate that incompleteness shocks can have significant effects in the allocation of capital across sectors and in aggregate output. In our benchmark calibration, an incompleteness shock can be associated with a relocation of more than 15% of total capital.

Related Literature This paper contributes to the well-developed literature that studies the role of financial markets as an amplification and propagation mechanism and as an independent source of business cycle fluctuations, following Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999). Most papers in this literature make the assumption that aggregate shocks are not hedgeable ex-ante. In particular, Bernanke and Gertler (1989) and Bernanke, Gertler and Gilchrist (1999) restrict the possibility of hedging aggregate shocks, while Kiyotaki and Moore (1997) only considers unanticipated shocks. Our paper differs from the previous literature in that it is the lack of financial markets *ex-post* but not the lack of financial markets *ex-ante* what generates a recession. Our results imply that a theory of why natural holders of productive assets decide to endogenously expose themselves to aggregate shocks is not necessary for financial disruptions to cause a recession.

It is by now well-established that the inability to hedge aggregate shocks ex-ante is relevant for financial frictions to matter. Krishnamurthy (2003) explicitly shows that agents in Kiyotaki and Moore (1997) would like to hedge, but are prevented from doing so. He shows that the collateral amplification mechanism is not robust to the introduction of markets that allow firms to hedge against common shocks and concludes that a model of the collateral amplification mechanism must incorporate a theory of incomplete hedging.[2] Kocherlakota (2000), Cordoba and Ripoll (2004), and Cole (2011) rely on this argument in various forms. In particular, Carlstrom, Fuerst and Paustian (2016) derive the optimal lending contract in the financial accelerator model of Bernanke, Gertler and Gilchrist (1999) and find that the privately optimal contract fully dampens the financial accelerator, while Di Tella (2017) shows that ex-ante hedging eliminates macroeconomic fluctuations in the model of Brunnermeier and Sannikov (2014).

Our work is most closely related to the work in which shocks to financial conditions are the direct drivers of business cycles. The work of Jermann and Quadrini (2012), who study financial shocks, defined as shocks that vary the ability of borrowers to raise funds, is perhaps closest. A financial shock in their model is a tightening of firms' borrowing constraint.[3] In our model, agents will be able to borrow and lend even after a financial shock, but they won't be able to hedge risks. The work by Mendoza (2010) and Jeanne and Korinek (2010) also features occasionally binding constraints. Importantly, these papers, as well as Buera and Moll (2014) and Buera and Nicolini (2015), who study the effects of credit frictions in the presence of heterogeneous producers, do not allow for ex-ante hedging markets. To our knowledge, we provide the first formulation with anticipated and hedgeable

[2]For instance, Rampini and Viswanathan (2010) provide a fully micro-founded model consistent with imperfect hedging in equilibrium.

[3]See also Perri and Quadrini (2017) for how shocks to enforcement constraints generate international co-movement.

financial shocks. Our framework is built around the idea of anticipated and hedgeable shocks, and to understand the importance of lack of financial markets, even when agents can borrow and lend freely. Both Christiano, Motto and Rostagno (2014) and Di Tella (2017) focus on changes in volatility, a real variable that pins down the quantity of risk, which gets amplified through financial imperfections. In our economy, the quantity of risk remains constant at all times. It is the inability to trade in financial markets that endogenously changes agent's valuations and endogenously affects real allocations.

Our model is formally close to the growing literature on continuous-time models of financial frictions, including He and Krishnamurthy (2012) and Brunnermeier and Sannikov (2014). We adopt instead a discrete-time formulation since it is more tractable for the type of shock that we consider. However, as those papers do, given that our argument crucially relies on unhedged risk exposures, we must solve our model using global methods. We use a time iteration procedure: see the detailed description of the numerical approach in the Appendix. Our results allow us to refine the conditions required for financial frictions to matter in those models. For instance, Brunnermeier and Sannikov (2014) qualify their results as follows:

> *" (...) instability in our model does depend on some aggregate risks being unhedgeable".*

We show that the crucial assumption for financial frictions to be relevant in their model and more generally is that some aggregate risks turn out to be unhedgeable ex-post, but not necessarily ex-ante.

Our model also relates to the literature that studies how the presence of adjustment costs affect equilibrium prices and aggregate quantities. See, for instance, Kogan (2001, 2004) and Eberly and Wang (2009, 2011). This body of work emphasizes the role of adjustment costs for capital allocation and asset prices in frictionless financial markets. In this paper, we assume that capital itself can be freely reallocated across sector to highlight the role of financing frictions. In practice, both adjustment costs and financial frictions are relevant for the allocation of capital.

Outline Section 2 describes the environment and Section 3 characterizes the equilibrium of the model and some of its properties. Section 4 illustrates the model mechanism through two examples and Section 5 solves and simulates the dynamic model. Section 6 discusses potential extensions and Section 6 concludes. The Appendix contains derivations and proofs. The Online Appendix provides computational details.

2 Model

We develop a parsimonious model of capital allocation and risk sharing in the presence of two types of aggregate shocks: technology shocks and incompleteness shocks.

2.1 Environment

Agents Time is discrete, with periods denoted by $t = 0, 1, 2, \ldots, T$, where $T = \infty$. When needed, we denote by $s^t \in S^t$ the history of events up to period t, given by $(s_0, s_1, \ldots, s_t)$, where $s_t \in S_t$ denotes one of finitely many events that may occur at period t.[4]

There are two goods, a single perishable consumption good, which serves as numéraire, and capital. There are two groups of agents, experts and households, each in unit measure and indexed by $i = \{E, H\}$. Both experts and households have identical preferences over consumption. Experts and households exclusively differ in the nature of the technology shocks that affect their ability to produce consumption goods using capital. We'll assume below that experts are more efficient than households at managing risky capital to produce consumption.

Both types of agents have recursive preferences of the Epstein-Zin form over consumption C_t^i, given by

$$U_t^i = \left[(1-\beta)\left(C_t^i\right)^{1-\frac{1}{\psi}} + \beta \mathbb{E}_t\left[\left(U_{t+1}^i\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{1}$$

for which the parameter $\gamma > 0$ corresponds to the relative risk aversion coefficient for static wealth gambles, while the parameter $\psi > 0$ corresponds to the elasticity of inter-temporal substitution (EIS) for a non-stochastic consumption path. The parameter $\beta \in (0,1)$ represents agents' discount factor. When $\gamma = \frac{1}{\psi}$, agents have expected utility of the CRRA type. Since agents' preferences satisfy an Inada condition, consumption is strictly positive in every period.

Production technology and market for capital Total capital in the economy, which does not depreciate, remains fixed at the level $\overline{K}$. Agents can trade capital every period in a spot market at a price Q_t. The timing of trades is as follows. A given agent enters period t with K_{t-1}^i units of capital. He then chooses to acquire $\Delta K_t^i = K_t^i - K_{t-1}^i$ units of capital, if ΔK_t^i is positive, or to sell, if ΔK_t^i is negative. As soon as capital is reshuffled, experts and households respectively produce $Z_t^E F\left(K_t^E\right)$ and $Z_t^H F\left(K_t^H\right)$ units of consumption good.

The production function $F(\cdot)$ is increasing, concave and satisfies and Inada condition. With little loss of generality, we assume that $F(\cdot)$ is homogeneous of degree α, so $F(k) = k^\alpha$, where $\alpha \in [0,1)$. The random variables $Z_t^i \geq 1$, which capture the productivity of experts and households managing capital, follow an exogenously determined Markov process. We refer to Z_t^i as technology shocks. We describe below how to define the shocks Z_t^i so that they feature an aggregate component and a reallocation component.

After production takes place, payments from the spot market for capital are settled, and agents consume. Therefore, we can formally express the budget constraints for experts and households as follows

$$C_t^i \leq Z_t^i F\left(K_t^i\right) - Q_t \Delta K_t^i + A_t^i, \tag{2}$$

where agent i's consumption must be covered by the output of production, net capital sales/purchases, and changes in the agents' net asset positions, denoted by A_t^E and A_t^H, which are described next. If

[4] To simplify the exposition, we use the notation C_t^i, as opposed to $C_t^i\left(s^t\right)$, for most variables. We use the more explicit notation exclusively for those financial variables for which specifying the history is relevant.

experts hold K_t^E units of capital, households must hold the remaining $\overline{K} - K_t^E$ units. Formally, market clearing for capital implies that

$$K_t^E + K_t^H = \overline{K}, \quad \forall t. \tag{3}$$

Financial markets The ability of financial markets to allocate wealth across periods and states determines the specific form of A_t^i. For both experts and households, A_t^i takes the form

$$A_t^i = B_{t-1}^i - \phi_t \sum_{s_{t+1}|s_t} q_t\left(s_{t+1}|\, s_t\right) B_t^i\left(s_{t+1}|\, s_t\right) - \left(1-\phi_t\right) \kappa q_t^f B_{t,f}^i$$

where $B_{t-1}^i = \phi_{t-1} B_{t-1}^i\left(s_t|\, s_{t-1}\right) + \left(1-\phi_{t-1}\right) \kappa B_{t-1,f}^i$ denotes the financial transfer received by type agents i at period t.[5] We denote by $s_t = \left\{Z_t^E, Z_t^H, \phi_t\right\}$ the set of exogenous state variables. The random variable $\phi_t \in \{0,1\}$, which captures the changes in the set of financial markets available to agents, follows an exogenously determined two-state Markov chain.

When $\phi_t = 1$, agents have access to a complete set of one-period Arrow-Debreu securities at period t. In that case, we say that markets are complete in period t. When $\phi_t = 0$, agents only have access to a single riskless bond that matures in one-period, if $\kappa = 1$, or they lack access to financial markets, if $\kappa = 0$. When $\phi_t = 0$, we say that markets are incomplete in period t. Importantly, our baseline formulation implies that agents can write contingent contracts on the realization of technological shocks and incompleteness shocks, as long as $\phi_t = 1$.

When markets are complete ($\phi_t = 1$), we use $B_t^i\left(s_{t+1}|\, s_t\right)$ to denote the number of one-period ahead Arrow-Debreu securities purchased by agents of type i at state s_t that pay at state s_{t+1}. The price of those securities is $q_t\left(s_{t+1}|\, s_t\right)$. When markets are incomplete ($\phi_t = 0$), we use $B_{t,f}^i$ to denote the number of one-period riskless bonds purchased by agents of type i at period t, which pay at period $t+1$. The gross riskless interest rate corresponds to $R_t^f = \frac{1}{q_t^f}$. Hence, the variable B_{t-1}^i corresponds to the payoff of the Arrow-Debreu security at period t if markets were complete at period $t-1$. If markets were incomplete at period $t-1$, B_{t-1}^i corresponds instead to the payoff of the riskless bond.

Financial assets are in zero net supply, so market clearing guarantees at every period t that

$$B_t^E\left(s_{t+1}|\, s_t\right) + B_t^H\left(s_{t+1}|\, s_t\right) = 0, \quad \forall s_{t+1}, \quad \text{if} \quad \phi_t = 1 \tag{4}$$

$$B_{t,f}^E + B_{t,f}^H = 0, \qquad \text{if} \quad \phi_t = 0 \text{ and } \kappa = 1. \tag{5}$$

This concludes our description of the environment. Summing up, there are two types of aggregate shocks in this economy: technology shocks, Z_t^i, and incompleteness shocks ϕ_t. Importantly, when $\phi_t = 1$, markets are complete in period t and agents are able to hedge both types of shocks, so incompleteness shocks are fully anticipated and hedgeable ex-ante in the sense that agents can arrange state contingent transfers of wealth towards states in which in a incompleteness shock occurs. Formally, we define an incompleteness shock as follows.

[5] We follow the convention that all endogenous variables determined after the realization of the period t shock and before the realization of the $t+1$ shock carry the subscript t.

Definition. (Incompleteness shock) We say that the economy experiences an incompleteness shock at period t when $\phi_t = 0$ and $\phi_{t-1} = 1$. In that case the economy switches from having well-functioning financial markets to being unable to share risks looking forward.

We do not take a stance on exact microeconomic nature of incompleteness shocks.[6] We interpret an incompleteness shock as any shock that impairs the ability of financial markets to share and transfer risks. These shocks are meant to capture the disruption in financial trading associated with a financial crisis, in which intermediary activity stalls and trading freezes are common. In our leading case in which $\kappa = 1$, agents still have access to borrowing and lending after an incompleteness shocks.

Our definition of competitive equilibrium in sequence form is standard. In the Appendix, we express the model and its equilibrium in recursive form, using the agent's wealth distribution as the single endogenous variable. The presence of incompleteness shocks prevents us from obtaining the equilibrium allocation by solving a planning problem and then decentralizing the allocation using prices.

Definition. (Competitive Equilibrium) A competitive equilibrium is characterized by a sequence of consumption, capital holdings, and asset holdings for each type of agent C_t^i, K_t^i, $B_t^i\left(s_{t+1}|\, s_t\right)$ if $\phi_t = 1$ or $B_{t,f}^i$ if $\phi = 0$ and $\kappa = 1$; capital prices Q_t and Arrow-Debreu prices $q_t\left(s_{t+1}|\, s_t\right)$ if $\phi_t = 1$ or riskless bond prices q_t^f if $\phi = 0$ and $\kappa = 1$ such that: i) agents choose consumption, capital holdings, and asset holdings to maximize utility (1) subject to constraints (2) and (3) and ii) the market for capital and all financial markets clear, that is, Equations (3), (4), and (5) hold.

Our formulation allows us to study the effects of incompleteness shocks. In particular, we can study how an economy reacts at impact when an incompleteness shock materializes and we can also study how an economy reacts in anticipation of an incompleteness shock. Importantly, we are able to address the question of whether having the ability to arrange the transfer of resources on a contingent basis towards states in which incompleteness shocks materializes exacerbate or mitigate its impact. At times, we compare the predictions of our economy with simpler benchmarks: i) an incomplete markets model in which only capital is traded, ii) an incomplete markets model in which capital and a single non-contingent bond are traded, and iii) a model in which markets are complete in each period.

2.2 Remarks on modeling assumptions

It's worth highlighting several of our modeling assumptions.

a. Structure of financial markets. The presence of incompleteness shocks breaks down the equivalence between sequential trading and time-0 trading common in standard complete market models; this equivalence is explained in Mas-Colell, Whinston and Green (1995) and Ljungqvist and Sargent (2004), among others. It is important for our results that agents in our economy only have access to one-period

[6] In addition to our references in the literature review, see Cooley, Marimon and Quadrini (2004) and Rampini and Viswanathan (2010) for papers than endogenize the form of the financing restrictions.

contracts. If agents were allowed to sign all possible long-term contracts, our model collapses to a complete markets benchmark, and financial shocks would be irrelevant for the allocation of capital and output. The one-period nature of contracts in our model is a tractable simplification. The key assumption needed for our arguments to be valid is that the there are states in which some risks are not hedgeable looking-forward.[7] Importantly, incompleteness shocks are anticipated and can be hedged ex-ante. That is, experts and households can arrange contingent transfers of resources conditional on the realization of the incompleteness shock.

b. Timing. Our assumption regarding the timing of production, allowing investors to trade before producing within the period, is important to simplify the solution of the model, as well as to find tractable analytical results, including the characterization of the production efficiency benchmark as well as the link between market incompleteness and equilibrium capital allocations. Had we assumed instead that agents trade capital at the end of the period subject to uncertainty about the productivity of capital does not significantly affect our insights. Both formulations only differ in the period t cash flow, and in both formulations the future price Q_{t+1} is a function of all remaining future cash flows.

c. Technology and preferences. Since our focus is on the allocation of capital across sectors and our narrative is focused on business cycles, rather than in long-run growth questions, we abstract from endogenous capital accumulation in the aggregate.[8] This assumption reduces the number of state variables but could be incorporated to our simulations at some cost. Also, we adopt decreasing returns to scale production functions and abstract from adjustment costs to capital, in order to yield a well-defined interior solution for capital and a clear first-best production benchmark.[9] We discuss in Section 6 how to introduce adjustment costs. To further highlight the role of incompleteness shocks, we make agents preferences symmetric. It is well-known that heterogeneity in preferences parameters (risk aversion, elasticity of inter-temporal substitution, and discount factors) can generate richer dynamics. It seems reasonable to consider those extensions in future work.

3 Equilibrium characterization

First, we establish the conditions for production efficiency and characterize the behavior of the economy under complete markets. Next, we study the equilibrium allocation of capital and aggregate output in the presence of incompleteness shocks.

[7] From a pure modeling perspective, it is easy to make reasonable assumptions that restrict the contracts to be short term. For instance, we can think of this economy as a succession of overlapping generations with bequest motives in which future generations can renege on promises made by past generations. Alternatively, one can argue that contract enforcement is less effective at long-horizons. In a different context, Angeletos (2002), Buera and Nicolini (2004), and Lustig, Sleet and Yeltekin (2008) emphasize the ability of long-term assets for hedging.

[8] Early work relating market completeness and growth includes Greenwood and Jovanovic (1990) and Acemoglu and Zilibotti (1997).

[9] Existing literature often relies on constant returns to scale and homogeneity to guarantee that the share of capital held by one of the agents is the relevant state variable.

3.1 Production efficiency and complete markets

For a given realization of technology shocks Z_t^E and Z_t^H, we define aggregate output Y_t in this economy as the sum of consumption goods produced by both experts and households, that is,

$$Y_t = \sum_{i=\{E,H\}} Z_t^i F\left(K_t^i\right).$$

The production efficiency benchmark is characterized by the solution to the planning problem $\max_{K_t^E, K_t^H} Y_t$, subject to the resource constraint for capital in Equation (3), $\sum_i K_t^i = \overline{K}$. Because agents' production technologies have decreasing returns to scale, this planning problem is well defined and features a unique interior optimum. At the optimum, the following condition must be satisfied

$$Z_t^E F'\left(K_t^E\right) = Z_t^H F'\left(K_t^H\right). \tag{6}$$

When Equation (6) holds at period t we say that the economy satisfies production efficiency. We characterize the allocation of capital and output in that case in the following Lemma.

Lemma 1. (Production efficiency benchmark) *When the economy satisfies production efficiency at period t,*

a) the allocation of capital across sectors is given by

$$K_t^{i\star} = w_t^{i\star}\overline{K}, \quad where \quad w_t^{i\star} = \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}} \quad and \quad \overline{Z}_t = \left(\sum_i \left(Z_t^i\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}, \tag{7}$$

b) aggregate output is given by

$$Y_t = \overline{Z}_t \overline{K}^\alpha, \quad where \quad \overline{Z}_t = \left(\sum_i \left(Z_t^i\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha},$$

c) while individual output is given by

$$Y_t^i = w_t^{i\star}\overline{Z}_t\overline{K}^\alpha.$$

From Equation (7), it is clear that any multiplicative aggregate shift of technology Z_t^i does not affect the relative allocation of capital across sectors. The share of capital allocated to type i agents as well as the share of total output produced by type i agents is determined by the weight $w_t^{i\star}$, which captures the relative technological differences between agents. Formally, $w_t^{i\star}$ is increasing in Z_t^i and satisfies $\lim_{Z_t^i \to \infty} w_t^{i\star} = 1$. When agents have access to identical production technologies, $w_t^{i\star} = \frac{1}{2}$, implying that $\overline{Z}_t = 2^{1-\alpha} Z_t$. More generally, one can show that $\overline{Z}_t$ is a decreasing function of α. When $\alpha \to 1$, the economy features almost constant returns to scale, and most capital is allocated to the most productive agent. When $\alpha \to 0$, dispersion in individual productivities is beneficial to counteract the effect of decreasing returns.

The marginal product of capital, equalized in equilibrium across agents, can be recovered from aggregates, since

$$Z_t^i F'\left(K_t^{i\star}\right) = \alpha \overline{Z}_t \overline{K}^{\alpha-1}.$$

In the following Lemma, we show that complete markets in a given period is a sufficient condition for production efficiency. This lemma provides a clear efficiency benchmark that facilitates the study of incompleteness shocks.

Lemma 2. (Production efficiency in period t under complete markets) *When markets are complete at period t, that is, $\phi_t = 1$, the economy satisfies production efficiency in that period.*

The reasoning behind this result will become evident after we characterize the equilibrium more generally. In our model, the fact that complete markets at period t guarantee that the continuation value of capital for both groups of agents is the same, since they can frictionlessly re-trade capital at the beginning of the next period, implies that their marginal productive in period t must be equalized, guaranteeing production efficiency. In addition to the ability of re-trade capital, this result hinges on the absence of adjustment costs. This result allows us to clearly highlight the effects of market incompleteness on the allocation of capital.

While production efficiency trivially follows when markets are always complete, i.e., when there are no incompleteness shocks, nothing guarantees the same outcome in an economy subject to incompleteness shocks. Lemma 2 shows that in our setup, completeness at period t is sufficient to induce production efficiency, even when the economy is potentially hit by incompleteness shocks in the future. On the contrary, when markets are complete in period t, the price of capital Q_t does depend on the distribution of wealth across agents, since the future price of capital will depend on the distribution of wealth after an incompleteness shock.

Finally, note that the economy aggregates to a representative agent under complete markets, and that equilibrium prices behave as in standard consumption based asset pricing models, where $C_t = Y_t = \overline{Z}_t \overline{K}^{\alpha}$.

3.2 Equilibrium

Given the symmetry in agents' preferences and technologies, the optimality condition for capital holdings for both experts and households is given by

$$Q_t = Z_t^i F'\left(K_t^i\right) + \mathbb{E}_t\left[m_{t+1}^i Q_{t+1}\right], \quad i = \{E, H\}, \tag{8}$$

where $m_{t+1}^i = \beta \left(\frac{V_{t+1}^i}{\zeta_t^i}\right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\frac{1}{\psi}}$ denotes the stochastic discount factor of type i agents, where we define the certainty equivalent of future continuation utility as $\zeta_t^i \equiv \mathbb{E}_t\left[\left(V_{t+1}^i\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$.[10] Intuitively,

[10]Alternatively, we can express m_{t+1}^i as follows: $m_{t+1}^i = \beta \left(\frac{\frac{V_{t+1}^E}{C_{t+1}^E}}{\frac{Y_t}{C_t^E}}\right)^{\frac{1}{\psi}-\gamma} \left(\frac{C_{t+1}^E}{C_t^E}\right)^{-\gamma}$.

agents equalize the marginal cost and benefit of holding capital at the margin. The marginal cost of holding capital is given by its price Q_t, while the marginal benefit of holding capital is given by its period t payoff, given by $Z_t F'\left(K_t^i\right)$ and its continuation value, given by $\mathbb{E}_t\left[m_{t+1}^i Q_{t+1}\right]$.

When $\phi_t = 1$, there are as many optimality conditions for every Arrow-Debreu security as possible shocks in the next period. Formally, the optimality condition for trading in financial markets by both experts and households satisfy

$$q_t\left(s_{t+1}|\,s_t\right) = \pi\left(s_{t+1}|\,s_t\right) m_{t+1}^i\left(s_{t+1}|\,s_t\right), \forall s_{t+1}, \quad i = \{E, H\}. \tag{9}$$

When $\phi_t = 0$ and $\kappa = 1$, both experts and households choose their holdings of the riskless asset optimally. Formally,

$$q_t^f = \mathbb{E}_t\left[m_{t+1}^i\right], \quad i = \{E, H\}. \tag{10}$$

Equations (8), (9), and (10), when combined with market clearing conditions (3) and (4), are sufficient to characterize the equilibrium of the economy for given initial conditions.

By combining experts' and households' Euler equations for capital, we can characterize the effect of financial market shocks on the allocation of capital in the economy and its impact on aggregate output. The main insights of the paper can drawn from Equation (11):

$$0 = Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right) + \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right], \tag{11}$$

which provides a clear link between distortions in financial markets and the allocation of capital in the economy. First, note that if markets are complete in period t, the allocation of capital in the economy is characterized by

$$Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right) = 0.$$

Therefore, although agents agree on the possible realizations of future prices Q_{t+1}, the possibility of entertaining different valuations for the asset will determine the allocation of capital in period t.

At any competitive equilibrium in which production efficiency does not hold, but capital can be freely traded, it must be that agents have different valuations for the continuation value of capital. If experts hold less capital than at the first-best, it must be that the continuation value than experts attach to capital is less than the continuation value of households. Intuitively, even though transferring a unit of capital from experts to households would increase aggregate production, the fact that experts would like to borrow today against the future or that capital is risky, make them more willing to hold more capital than at the optimum.

Intuitively, if $Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right) > (<)\, 0$, the (social) marginal value of reallocating a marginal unit of capital from experts to households is positive (negative). However, this may not occur in the decentralized equilibrium whenever $\mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] < (>) 0$, that is, when the continuation value of the asset for experts is less (higher) than for households. That is

$$Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right) > 0 \iff \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] < 0 \Rightarrow K_t^E < K_t^{E\star}$$
$$Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right) < 0 \iff \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] > 0 \Rightarrow K_t^E > K_t^{E\star},$$

where $K_t^{E\star}$ corresponds to the allocation of capital under production efficiency, defined in Lemma 1.

We formalize and expand on this logic in Proposition 1, by formally defining an incompleteness wedge Ω. Any non-zero wedge is associated with capital misallocation. Our results rely on the dual role of capital as a productive input as well as a durable store of value. When financial markets are well-functioning, both roles can be decoupled. Otherwise, the degree of financial frictions will affect the allocation of capital.

Proposition 1. (Incompleteness wedge and capital allocation) *a) The difference in the valuation of capital between experts and households using their respective stochastic discount factors, which we refer to as the incompleteness wedge and define by Ω_t, is a sufficient statistic for the allocation of capital between experts and households in this economy. Formally, we define Ω_t as*

$$\Omega_t \equiv \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right], \quad \textit{Incompleteness wedge.}$$

When $\Omega_t > 0$, experts hold too little capital relative to the production efficiency benchmark; households hold too much capital in that case. When $\Omega_t < 0$, experts hold too much capital relative to the production efficiency benchmark; households hold too little capital in that case.

b) Moreover, the variable Ω_t can be decomposed in a risk-free wedge and the risky wedge. Formally

$$\Omega_t = \Omega_t^f + \Omega_t^r, \quad \textit{where}$$

$$\Omega_t^f \equiv \left(\mathbb{E}_t\left[m_{t+1}^H\right] - \mathbb{E}_t\left[m_{t+1}^E\right]\right)\mathbb{E}_t Q_{t+1}, \qquad \textit{Risk-free wedge,}$$
$$\Omega_t^r \equiv \mathbb{C}ov\left[m_{t+1}^H, Q_{t+1}\right] - \mathbb{C}ov\left[m_{t+1}^E, Q_{t+1}\right], \quad \textit{Risky wedge.}$$

If $\Omega_t > 0$, the continuation value of holding the asset is higher for households than for experts, so households hold more capital in equilibrium than at the production efficiency benchmark. The variable Ω_t has two components. The first component Ω_t^f corresponds to the purely inter-temporal part of the SDF. The term Ω_t^f directly depends on the difference across agents in shadow risk free rates:

$$\mathbb{E}_t\left[m_{t+1}^H\right] - \mathbb{E}_t\left[m_{t+1}^E\right] = \frac{1}{R_t^{f,H}} - \frac{1}{R_t^{f,E}}$$

If $\Omega_t^f > 0$, experts would like to borrow and households would like to save or, alternatively, the relative investment opportunities of experts are relatively better on average at date t. This type of distortion is of the same nature as the distortion that arises in models with binding borrowing constraints. When agents can borrow and save through non-contingent financial instruments without frictions, this term is zero. For instance, when $\kappa = 1$, given that a riskless bond is available for trading after an incompleteness shocks $\Omega_t^f = 0$. When $\kappa = 0$, we allow for Ω_t^f to be different from zero. When $\Omega_t^f = 0$, any distortion associated with the allocation of capital can be traced back to how the ability to trade in financial markets affects the hedging properties of capital.

The second component Ω_t^r arises because of the inability to hedge risks across different states, even when on average households and experts could equalize expected marginal utilities. Formally, Ω_t^r can be written as $\mathbb{C}ov\left[m_{t+1}^H, Q_{t+1}\right] - \mathbb{C}ov\left[m_{t+1}^E, Q_{t+1}\right]$. If $\Omega_t^r > 0$, capital is relatively riskier for experts than for households, so we expect experts to hold less capital than at the production efficiency benchmark. Under the presumption that experts are the natural holders of aggregate risk, we expect their stochastic discount factor to be more correlated with Q_{t+1}, making $\Omega_t^r > 0$ the natural case when an incompleteness shock occurs. On the other hand, if $\Omega_t^r < 0$, capital is a relatively better hedge for experts. In that case, K_t^E is relatively higher than the production efficiency benchmark.

While previous literature has highlighted that restrictions on borrowing and lending are a key determinant of how capital gets allocated, much less attention has been given to the absence of hedging markets. Consistent with Lemma 1, when $\phi_t = 1$, markets are complete in period t, guaranteeing the existence of a unique stochastic discount factor, which implies that $m_{t+1}^E = m_{t+1}^H$. In that case, $\Omega_t^f = \Omega_t^f = 0$ and, consequently, $\Omega_t = 0$, so production efficiency holds. Note also that when $\beta \to 0$, the economy converges to production efficiency.

After establishing the link between the incompleteness wedge and the allocation of capital, we can study how incompleteness affects aggregate output.

Proposition 2. (Misallocation with non-negative incompleteness wedge) *The (endogenous) aggregate TFP of the economy is a negative function of the incompleteness wedge Ω_t. Formally, aggregate output can be expressed as*

$$Y_t = \Xi\left(\Omega_t\right) \overline{Z}_t \overline{K}^{\alpha}, \tag{12}$$

where $\Xi\left(\Omega_t\right) = \sum_i \left(\frac{\tilde{Z}_t^i/\tilde{Z}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} w_t^{i\star}$ is a decreasing function of $|\Omega_t|$.

Proposition 2 relates the presence of financial frictions to the level of aggregate output. Departures from complete markets will be associated with a non-zero incompleteness wedge. Given the concavity of agents' production technologies, which defines a single optimal allocation of capital across sectors, any deviation from the the production efficiency benchmark characterized in Lemma 1 will be associated with an output loss.

This result is consistent with the work by Moll (2014), Buera and Moll (2014), and David, Hopenhayn and Venkateswaran (2016), who show that financial frictions in the presence of heterogeneity in the productivity of final goods producers is associated with an efficiency loss.

Finally, in Proposition 3 we address the question of whether ex-ante hedging against incompleteness shocks will be able to mute their effect. The answer is negative.

Proposition 3. *(Ex-post vs ex-ante incompleteness) The allocation of capital across agents and aggregate output will be generically inefficient in period t whenever an incompleteness shocks hits the economy, even when transfers can be arranged ex-ante contingent on the realization of the incompleteness shock.*

From Proposition 1, it is clear that as long as agents cannot equalize their marginal rates of substitution across all states, there will be capital misallocation in period t. It should also be clear that freely arranging the set of relevant state variables in period t cannot eliminate the capital misallocation in general. Before the incompleteness shock, Equation (9) will be valid, and agents will find optimal to equalize their marginal utilities, which can ameliorate the impact of shocks. However, there is no guarantee that reallocating resources among agents in period t is sufficient to equalize their marginal rates of substitution across periods and states in the future.

The general insight is that, from the perspective of period t, any restriction regarding the risk-sharing of *future* risks is sufficient to distort the allocation of capital at date t, independently of the availability of ex-ante hedging opportunities. This logic becomes more salient when agents have access to free borrowing and lending after an incompleteness shock. In that case, agents are unconstrained transferring resources between period t and the future. However, in that case, the inability to hedge future risks is sufficient to generate capital misallocation.

4 Inspecting the mechanism

Before solving the dynamic model, we illustrate Propositions 1 through 3 with two simple three-period applications of the general framework. Both scenarios highlight how market incompleteness affects the allocation of capital through a) the inability to borrow/save across periods and b) the absence of markets to hedge risks.

In both scenarios, we assume that $T = 2$, so time runs $t = 0, 1, 2$. We also assume that $\gamma = \frac{1}{\psi}$, so agents have time separable CRRA utility $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

4.1 Scenario 1: risk-free wedge ($\kappa = 0$)

Let us start with the case where there is no risk free bond. The only long term asset available to the agent is productive capital. As a result, saving and production decisions become inter-twined and the equilibrium is (typically) not efficient.

In this scenario, we model technology shocks as follows. Agents' technology is defined as $Z_t^i = Z_t \theta_t^i$, where we assume that the common technology component is constant and normalized to $Z_t = 1$ for all periods. We describe below the equilibrium of the economy for different combinations of θ_0^E, θ_1^E, and θ_2^E, which map one-to-one to changes in the efficient level of asset holdings by experts in the first-best allocation, as we explained in Section B of the Online Appendix. This formulation implies that aggregate output is constant if production efficiency holds every period. We also assume that agents' initial capital holdings corresponds to $K_{-1}^i = w_0^{i\star}\overline{K}$, which will imply that no trade in capital is optimal in period 0. Although it is not necessary, we can assume that the experts hold always more than half of the capital stock in the frictionless benchmark.

Figure 1 illustrates the pattern of market incompleteness. Markets are trivially complete in period 2, since there is no residual uncertainty. Markets are also complete in period 0, since agents can trade a risk-free bond and the state of nature in period 1 is known with certainty. On the contrary, in period 1, the economy is hit with an incompleteness shocks in which agents lack access to financial markets, although they can still trade capital freely (we set $\kappa = 0$). In this economy, even though agents can trade capital, which is a long-lived asset, they will not be able to equalize their marginal rates of substitution between periods 1 and 2 in equilibrium, since capital plays a dual role as a production input and a store of value.

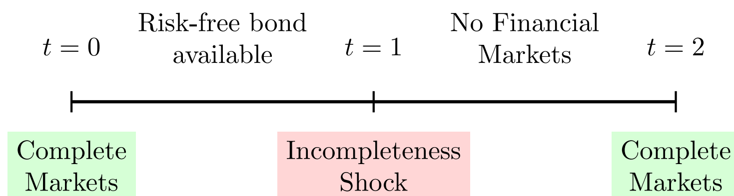


Figure 1: Market structure in scenario 1

Therefore, we can express the problem solved by agents as

$$\max U\left(C_0^i\right) + \beta U\left(C_1^i\right) + \beta^2 U\left(C_2^i\right),$$

while their budget constraints correspond to

$$\begin{aligned} C_0^i &= Z_0^i F\left(K_0^i\right) - Q_0 \Delta K_0^i - \frac{B_1^i}{R_1} \\ C_1^i &= Z_1^i F\left(K_1^i\right) - Q_1 \Delta K_1^i + B_1^i \\ C_2^i &= Z_2^i F\left(K_2^i\right) - Q_2 \Delta K_2^i. \end{aligned}$$

Our assumptions make the characterization of the equilibrium relatively simple. At date 2, markets are effectively complete, and production efficiency holds. In that case, $K_2^i = w_2^{i\star}\overline{K}$, where $w_2^{i\star} = \left(\frac{Z_2^i}{\overline{Z}_2}\right)^{\frac{1}{1-\alpha}}$. The equilibrium price at date 2 is given by $Q_2 = \alpha \overline{Z}_2 \overline{K}^{\alpha-1}$ and aggregate output corresponds to $Y_2 = \alpha \overline{Z}_2 \overline{K}^{\alpha}$. We show in the Appendix that equilibrium consumption is thus given by

$$C_2^i = \left((1-\alpha)\, w_2^{i\star} + \alpha w_1^i\right) \overline{Z}_2 \overline{K}^{\alpha}, \tag{13}$$

where w_1^i is the share of capital held by type i agents in equilibrium. The equilibrium value of w_1^i is in principle different from the efficient $w_1^{i\star}$.

At date 1, markets are incomplete, and agents will not generically be able to equalize the marginal utility of consumption between periods 1 and 2. For a given level of B_1^i (determined in period 0), and equilibrium of the economy in period 1 is characterized by a pair of equations of the form

$$Q_1 = Z_1^i F'\left(K_1^i\right) + \beta \frac{U'\left(C_2^i\right)}{U'\left(C_1^i\right)} Q_2,$$

combined with the market clearing condition for capital, where Q_2 and C_2^i are described above, and C_{1i} is given by

$$C_1^i = \Xi_1^i w_1^{i\star} \overline{Z}_1 \overline{K}^{\alpha} - Q_1 \Delta K_1^i.$$

where $\Xi_1^i = \left(\frac{\tilde{Z}_t^i/\tilde{\overline{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}}$ is a decreasing function of Ω_1, as we show in the Appendix.

At date 0, we can exploit Lemma 1, combined with our assumption on initial capital holdings, to guarantee that $\Delta K_0^i = 0$, so $C_0^i = Z_0^i F\left(K_0^i\right) - \frac{B_1^i}{R_1}$. The absence of uncertainty allows us to characterize analytically the equilibrium level of savings for agent i as follows

$$\frac{B_1^i}{R_1} = \frac{\beta}{\beta + (\beta R_1)^{1-\frac{1}{\gamma}}} \left(Z_0^i F\left(K_0^i\right) - \frac{Z_1^i F\left(K_1^i\right) - Q_1 \Delta K_1^i}{\Xi\left(\Omega_1\right)} \right),$$

where $\Xi\left(\Omega_1\right) < 1$ corresponds to the misallocation loss and is defined as in Proposition 2. The equilibrium interest rate between periods 0 and 1 can be found from aggregates $R_1 = \frac{1}{\beta}\left(\frac{\Xi(\Omega_1)\overline{Z}_1\overline{K}^{\alpha}}{\overline{Z}_0\overline{K}^{\alpha}}\right)^{\gamma} = \frac{1}{\beta}\left(\Xi\left(\Omega_1\right)\right)^{\gamma}$. We characterize the properties of the equilibrium for different values of θ_0^E, θ_1^E, and θ_2^E in Proposition 4.

Proposition 4. *(Risk-free wedge) a) If $\theta_0^E = \theta_1^E = \theta_2^E$, the economy reaches its first-best allocation.*

b) If $\theta_0^E = \theta_1^E$ and $\theta_1^E \leq (\geq)\, \theta_2^E$, experts are more (less) productive in period 0. In that case, they find optimal to reduce (increase) their capital holdings at date 1 below (above) the first-best. Experts borrow (save) in equilibrium in period 0.

c) If $\theta_0^E \geq \theta_1^E$ and $\theta_1^E \leq \theta_2^E$, experts are less productive in period 1 worse off. In that case, they find optimal to over accumulate capital at date above the first-best level. Experts hold too much capital, $K_1^E \geq K_1^{E*}$. Experts save in equilibrium.

It should not be surprising that if agents' technology is identical and constant across periods the economy reaches the first-best. That economy effectively features complete markets, since there are no meaningful needs for inter-temporal smoothing.

When $\theta_0^E = \theta_1^E$ and $\theta_1^E \leq \theta_2^E$, experts anticipate in period 1 that they will be relatively more productive in period 2. Anticipating their higher productivity and output, experts would find optimal to borrow against that future increase in resources. The lack of financial markets prevents them from doing so and induces them instead to sell capital beyond the efficient level, which allows them to imperfectly equalize their consumption between periods 1 and 2. This logic seems valid for a given level of B_1^i, but couldn't experts ex-ante decide to save towards period 1 and avoid capital sales? On the contrary, perhaps surprisingly, in this scenario experts find optimal to borrow in period 0, which instead tends to exacerbate the need for selling capital in period 1. Intuitively, in period 0 households find optimal to borrow to increase their period 0 consumption, with the intention of enjoying a smoother consumption path.

The third combination of parameters that we consider is meant to capture of a crisis scenario, in which experts suffer a temporary negative technology shock. The behavior of experts between dates 1 and 2 is similar to the previous case. Experts find optimal to sell capital beyond the efficient level to be able to smooth the negative shock. However, in this case, experts decide to bet net savers between periods 0 and 1.

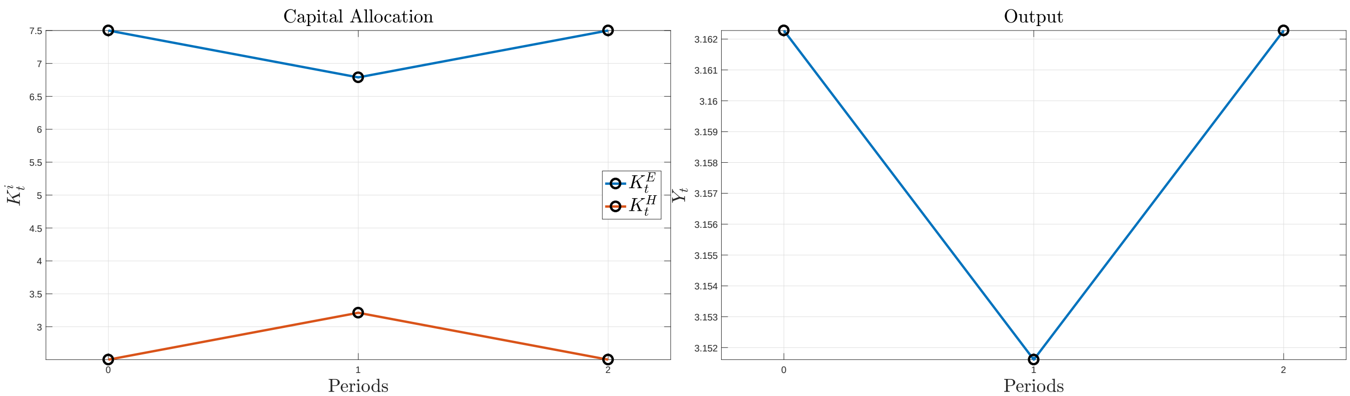


Figure 2: Capital Allocation and Output

Note: This figure illustrates Proposition 4c). It employs parameters $\theta_t^E = 0.75$, $\theta_t^E = 0.6$, and $\theta_t^E = 0.75$, as well as $\beta = 0.9$, $\gamma = 3$, $\alpha = 0.5$ and $\overline{K} = 10$.

These three parameter combinations highlight that the effects of the incompleteness vary non-trivially with the state of the economy.

4.2 Scenario 2: risky wedge ($\kappa = 1$)

In this scenario, we model technology shocks as follows. We assume that households technology is constant and normalized to $Z_t^H = 1$. We assume that experts' technology is weakly better $Z_t^E \geq 1$ than households' but random. In particular, we assume that $Z_0^E = Z_1^E$ and that in period two there are two possible realizations, such that $Z_{2U}^E > Z_{2D}^E$. This formulation of technology endogenously induces a correlation between aggregate output and experts' technology.

Figure 3 illustrates the pattern of market incompleteness. Market are trivially complete in period 2, since there is no residual uncertainty. Markets are also complete in period 0, since agents can trade a risk-free bond and the state of nature in period 1 is known with certainty. On the contrary, in period 1, the economy is hit with an incompleteness shocks in which agents only have access to a single non-contingent bond, that is, we set $\kappa = 1$. In this economy, because agents can trade the non-contingent bond in period 1, the risk-free wedge is necessarily zero.

Therefore, we can express the problem solved by agents as

$$\max U\left(C_0^i\right) + \beta U\left(C_1^i\right) + \beta^2 \mathbb{E}\left[U\left(C_2^i\right)\right],$$

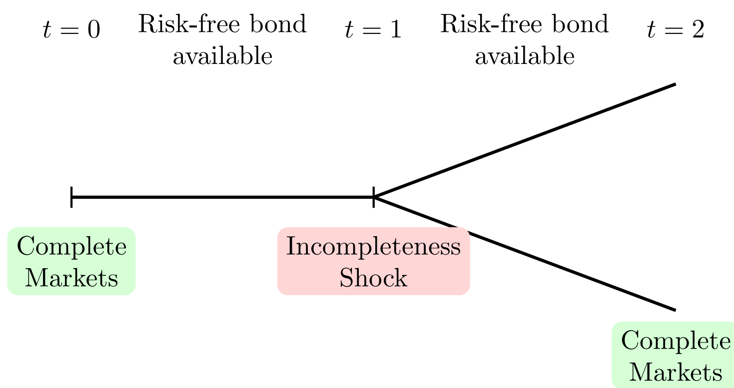


Figure 3: Market structure in scenario 2

while their budget constraints correspond to

$$
\begin{aligned}
C_0^i &= Z_0^i F\left(K_0^i\right) - Q_0 \Delta K_0^i - \frac{B_1^i}{R_1} \\
C_1^i &= Z_1^i F\left(K_1^i\right) - Q_1 \Delta K_1^i + B_1^i - \frac{B_2^i}{R_2} \\
C_2^i &= Z_2^i F\left(K_2^i\right) - Q_2 \Delta K_2^i + B_2^i.
\end{aligned}
$$

Once again, the characterization of the equilibrium is relatively simple. At date 2, markets are effectively complete, and production efficiency holds. In that case, $K_2^i = w_2^{i\star} \overline{K}$, where $w_2^{i\star} = \left(\frac{Z_2^i}{\overline{Z}_2}\right)^{\frac{1}{1-\alpha}}$. The equilibrium price at date 2 is given by $Q_2 = \alpha \overline{Z}_2 \overline{K}^{\alpha-1}$ and aggregate output corresponds to $Y_2 = \alpha \overline{Z}_2 \overline{K}^{\alpha}$. The same logic applies to period 0, in which markets are complete.

The characterization of the equilibrium is similar to the previous case.

Proposition 5. *(Risky wedge) In period* 1*, experts find optimal to reduce their capital holdings at date 1 below the first-best level. Experts save in equilibrium in period* 0*.*

Intuitively, the structure of the shocks makes experts consumption higher in states in which the date 2 price is higher, which makes the term $\mathbb{C}ov\left[m_{t+1}^E, Q_{t+1}\right]$ positive, as well as the whole risky wedge. Ex-ante, in anticipation of the incompleteness shock. Since the incompleteness shocks is relatively worse news for experts, experts find optimal to save between periods 0 and 1. These two scenarios allow us to provide better intuition for the results of the dynamic simulations.

5 Dynamic simulation

In this section, we present several numerical experiments whose focus is to illustrate the behavior of the economy under incompleteness shocks. As of now, our analysis is not meant to generate full-fledged quantitative answers, its focus is on illustrating the underlying economic mechanisms. We parametrize the model to generate sensible aggregate predictions and to highlight the impact of incompleteness shocks on capital reallocation and aggregate output.

To understand the implications of different forms of incompleteness, we study three different benchmarks. First, we consider a model in which agents can only trade capital. Formally, this model corresponds to setting $\phi_t = 0, \forall t$, and $\kappa = 0$. Second, we study a model in which agents are able to trade both capital and a single non-contingent bond. Formally, this model corresponds to setting $\phi_t = 0, \forall t$, and $\kappa = 1$. Next, we study the complete markets benchmark, which formally corresponds to setting $\phi_t^i = 1, \forall t$. Finally, we study how incompleteness shocks affect the equilibrium of the economy.

Parametrization We describe our choice of parameters in Table 1. The choices of discount factor, risk-aversion, and inter-temporal elasticity of substitution are all consistent with standard values in the asset pricing literature for a quarterly calibration. The curvature of technology is chosen to match the capital share, which is consistent with interpreting our technology specification as if other factors of production (e.g.: land, labor, etc.) were fixed.[11]

Table 1: Baseline parametrization

Parameter	Description	Values
β	Discount Factor	0.95
γ	Risk Aversion	3
ψ	Inter-temporal Substitution	1.5
α	Curvature of Production Technology	0.33
$\overline{K}$	Aggregate Capital	10
$1-\pi_0$	Probability of Incompleteness Shock	0.1
$1-\pi_1$	Probability of Recovery	0.5
σ_z	Volatility of Productivity Shock	0.075
ρ_z	Persistence of Productivity Shock	0.25

We approximate aggregate productivity shocks by a two-state Markov chain using Rouwenhorst's procedure. Formally, we seek to approximate a process of the form

$$Z_t^E = 1 + e^z, \qquad \text{with} \qquad z' = \rho_z z + \varepsilon' \qquad \text{and} \qquad \varepsilon \sim N\left(0, \sigma_\varepsilon^2\right).$$

We further renormalize Z_t^E so that its lower realization is 1. Formally, after approximating Z_t^E, we effectively solve the model for a Markov chain with transition between states $Z_t^E = 1$ and $Z_t^E = 1.46$ of

$$Z_t^E \sim \begin{array}{c} 1 \\ 1.4 \end{array} \left| \begin{pmatrix} 0.78 & 0.22 \\ 0.22 & 0.78 \end{pmatrix} \right. .$$

Note that the efficient capital allocations respectively correspond to $K^{E\star}\left(Z^E = 1\right) = 5$ and $K^{E\star}\left(Z^E = 1.46\right) = 5.39$.

[11] Current simulations, which correspond to $\beta = 0.8$, show similar qualitative behavior. Higher β amplifies departures from complete markets, since it emphasizes the forward-looking nature of the model.

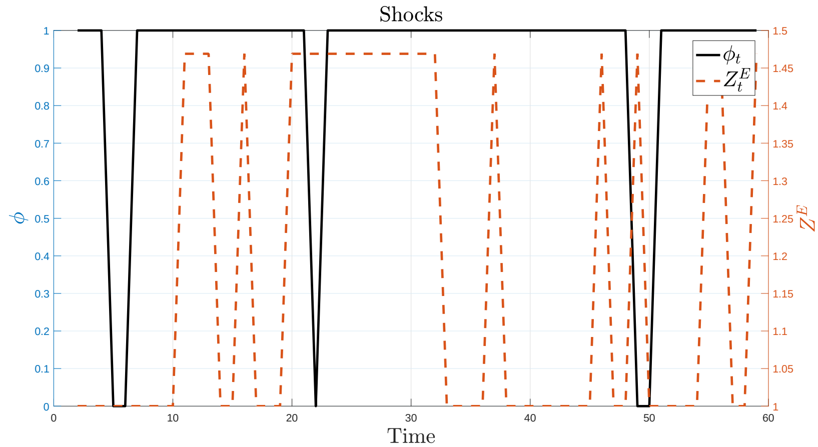


Figure 4: Realized shocks for the simulation

Note: This figure shows the realization of shocks used in all simulations.

We model incompleteness shocks as another independent two-state Markov chain. The probability of an incompleteness shock materializing, conditional on market working appropriately, is chosen to be 10%. The probability of recovery is 50%. Therefore, the evolution of ϕ_t is governed by the matrix

$$\phi_t \sim \begin{array}{c|c} 1 & \multirow{2}{*}{} \\ 0 & \end{array} \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}.$$

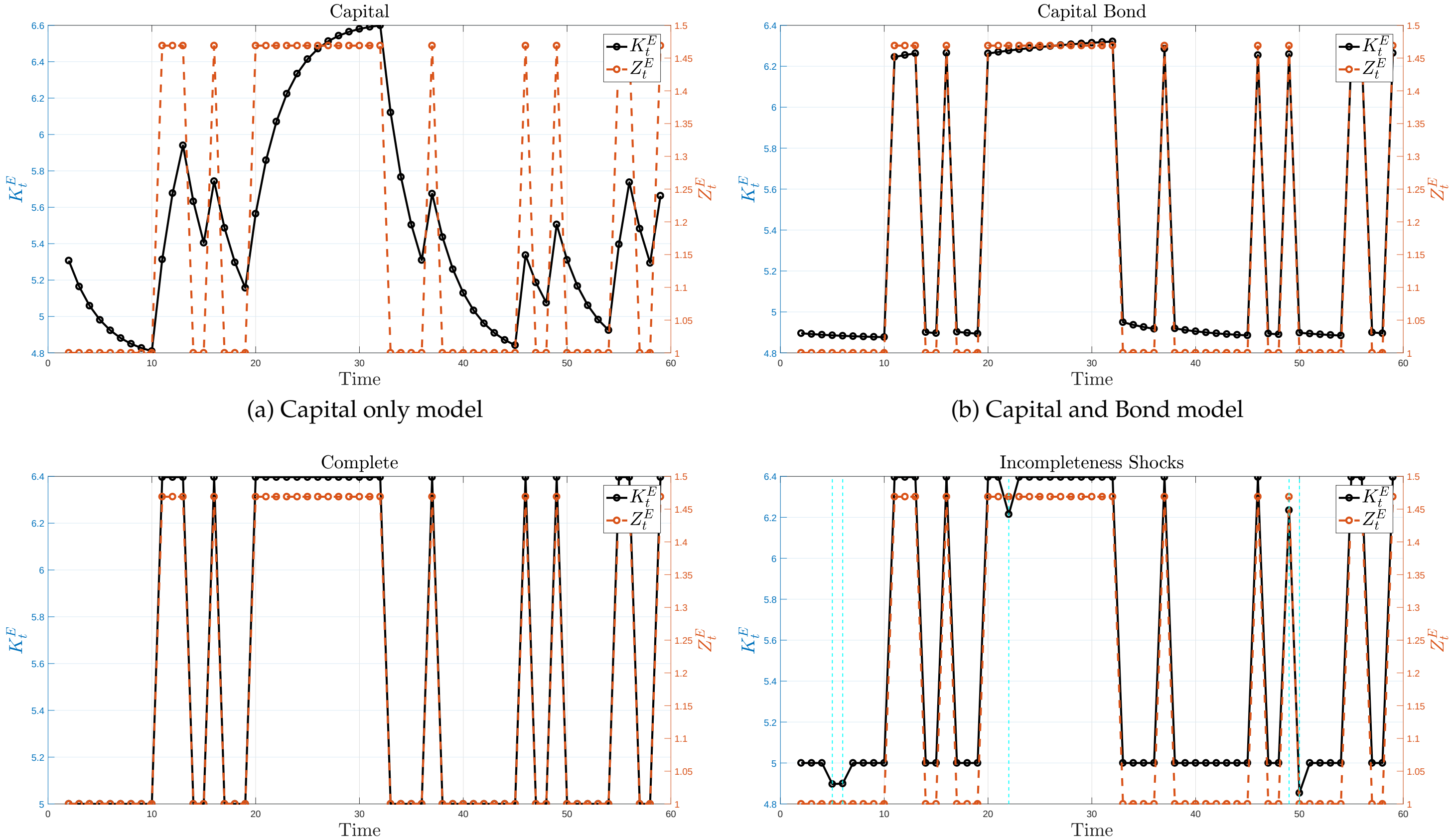
It's worth emphasizing that our model is tightly parametrized and has few degrees of freedom, in particular regarding the cross-sectional allocation of capital.

Because financial markets are at times incomplete, one must solve simultaneously for allocations and pricing functions. We find the solution of our model using global methods. Specifically, we use a time iteration/policy function iteration procedure, following the methods described in Coleman (1990), Judd (1998), Li and Stachurski (2014) and Rendahl (2015). Our appendix contains a detailed exposition of the numerical procedures followed. Provided initial guesses are reasonable, convergence is monotonic and reasonably fast. Solving the different benchmarks is conceptually straightforward. The presence of incompleteness shocks makes the solution of the model computationally burdensome.

The current simulations use grids for capital and bonds/Arrow-Debreu securities of $n_K = n_B = 20$. When markets are complete, we need to solve for $2 \times n_Z = 4$ Arrow-Debreu markets, to account for the possibility of hedging incompleteness shock. We approximate the B grids in an interval $[-5, 5]$. Current tolerance levels are 10^{-6} for all value and policy functions. All variables converge in less than 400 iterations.

Simulation Results We illustrate the behavior of the model through a simulation over $T = 60$ periods. We adopt $B_0^E = 0$ and $K^E = 5.5$ as starting values for all simulations. Figure 4 represents the realizations of the exogenous processes Z_t and ϕ_t. We purposefully chose a set of realizations in which incompleteness shocks materialize for both high and low technology realizations.

We compare the behavior of the economy for the same path of realizations of exogenous shocks for



(a) Capital only model

(b) Capital and Bond model

(c) Complete Markets model

(d) Incompleteness Shocks model

Note: All figures show the evolution of capital held by experts over time, which is one of the key state variables of the model. Figure (a) corresponds to the scenario in which $\phi_t = 0, \forall t$ and $\kappa = 0$. Figure (b) corresponds to the scenario in which $\phi_t = 0, \forall t$ and $\kappa = 1$. Figure (c) corresponds to the scenario in which $\phi_t = 1, \forall t$. Figure (d) corresponds to the scenario in which $\kappa = 1$ and ϕ_t is stochastic.
Vertical dashed turquoise lines represent the periods in which an incompleteness shock hits the economy.

Figure 5: Equilibrium evolution of experts' capital K_t^E

the different benchmarks. In Figure 5 we show the evolution of capital and technology for each of the four benchmarks. Figure 6 collects all series to highlight their differences. Several insights emerge from our simulations. In Figure 7 we show the evolution of asset holdings and technology for each of the four benchmarks.

Two features stand out from the model with only capital. First, the level of capital reacts slowly to technology shocks relatively to the first-best complete markets benchmark. This captures the effect of financial frictions slowing down the reallocation of capital across sectors. Second, for persistent series of positive technology shocks, experts tend to over-accumulate assets relative to the first-best. This behavior is due to the fact that agents use the capital good for self-insurance. Since experts can't save otherwise, the decide to inefficiently acquire and store capital as a form of savings. On the other hand, after a series of negative technology shocks, experts find optimal to hold less capital than at the first-best benchmark.

The evolution of capital is substantially different when agents can directly transfer resources intertemporally using the risk-free bond. In that case, we first observe that the response of capital to

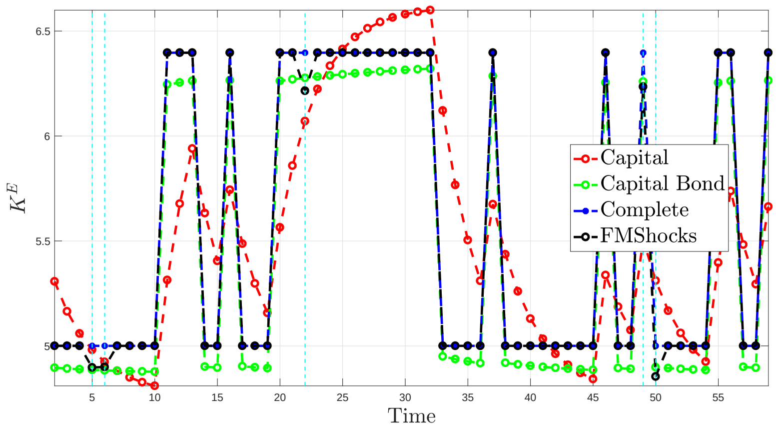


Figure 6: Capital evolution across formulations

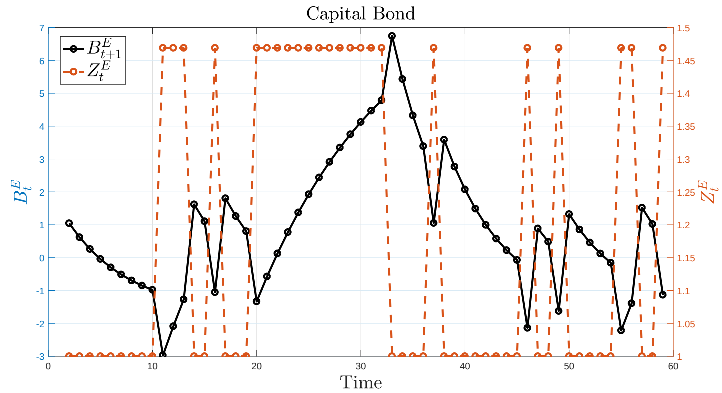
Note: This figure compares the evolution of capital across different formulation.

technology shocks is faster, since experts can borrow after positive shocks and lend after negative shocks – this is the behavior shown below in Figure 7(b). Interestingly, in this case we observe that even after a long series of positive shocks, experts hold now *less* capital relative to the first. This is due to the risky-wedge (using the terminology of Proposition 1), which is always higher for natural-holders/experts, since they hold always more capital.

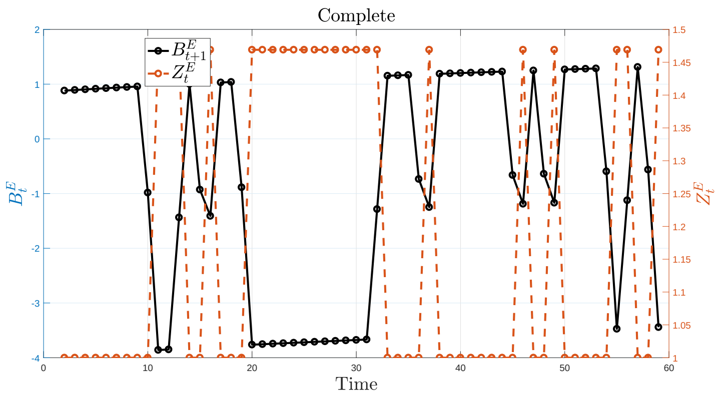
Capital in the complete markets benchmark satisfies production efficiency, as shown in Lemma 2. The production efficiency allocation extends to the general model whenever the economy is not hit by incompleteness shocks. The behavior of the economy after it is hit with an incompleteness shock, even though it is anticipated and fully hedgeable, involves experts holding less capital than otherwise. For both high and low realizations of the productivity shock, the incompleteness shocks induces experts to sell capital. Figure 7c) shows that experts partially save towards the incompleteness shock relatively more than they would, it is still not enough to eliminate the misallocation of capital, as described by Proposition 3.

Figure 8 shows the evolution of endogenous TFP, Ξ_t , as defined in proposition 2. Its noticeable that TFP falls more in periods in which the economy is in a high technology shock. This is natural and due to the fact that experts hold more capital in states when Z_t^E is high.

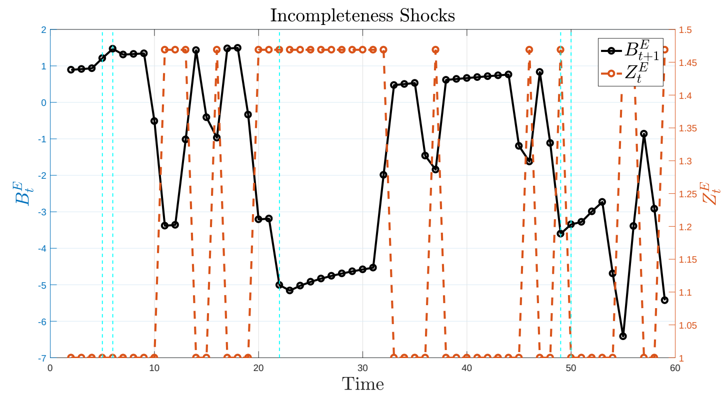
Sensitivity As expected from our analytical results, our results are highly sensitive to the choice of β. In particular, high values of β increase the importance of forward-looking effects, which amplifies the effects of market incompleteness and incompleteness shocks. In a preliminary exploration of the parameter space, it seems that high risk aversion also increase the effect of incompleteness shocks. The persistence of Z^E shocks is also a relevant parameter. Only for for intermediate values of ρ_Z incompleteness shocks have large effects. When shocks are too persistent, the experts self-insure very well using the risk-free bond. For the incompleteness wedge (in this case the risky wedge) to be positive, there must be some residual reallocation risk that cannot be self-insured with the bond.



(a) Capital and Bond model



(b) Complete Markets model



(c) Incompleteness Shocks model

Note: Note that in the model with only capital, financial asset holdings are trivially zero. Figure (a) corresponds to the scenario in which $\phi_t = 0, \forall t$ and $\kappa = 1$. It illustrates the level of borrowing/savings by experts. Figure (b) corresponds to the scenario in which $\phi_t = 1, \forall t$. This figure shows at period t the level of Arrow/Debreu savings/borrowing towards the future state realized (each agent has portfolio of as many securities as possible states, in this Figure we only show the one level of contingent borrowing/savings that materializes along the equilibrium path. Figure (c) shows either the level of noncontingent/borrowing and savings, under incompleteness shocks, or the choice of Arrow/Debreu securities bought/issued in equilibrium, which corresponds to the scenario in which $\kappa = 1$ and ϕ_t is stochastic.
Vertical dashed turquoise lines represent the periods in which an incompleteness shock hits the economy.

Figure 7: Equilibrium behavior of expert's asset holdings

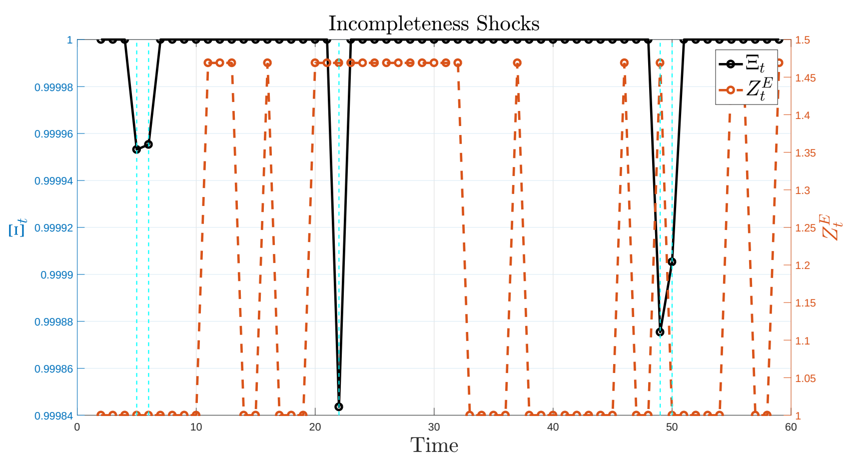


Figure 8: Endogenous TFP

Note: This figure shows the realization of shocks that we use for our simulations.

6 Extensions

In work in progress, we consider several extensions to the baseline model.

Adjustment costs It is conceptually easy to incorporate adjustment costs to the capital in our framework. Formally, we can assume that agents face a convex adjustment cost $h(\cdot)$, such that $h(0) = 0$, to changes in their capital stock, as in

$$C_t^i \leq Z_t^i F\left(K_t^i\right) - h\left(\Delta K_t^i\right) - Q_t \Delta K_t^i + A_t^i.$$

This formulation modifies agents' optimality condition for capital, which now corresponds to

$$Q_t = Z_t^i F'\left(K_t^i\right) - h'\left(\Delta K_t^i\right) + \mathbb{E}_t\left[m_{t+1}^i Q_{t+1}\right].$$

Intuitively, adjustment costs slow down the reallocation of capital. With adjustment costs, the production efficiency benchmark, characterized by

$$Z_t^E F'\left(K_t^E\right) - h'\left(\Delta K_t^E\right) = Z_t^H F'\left(K_t^H\right) - h'\left(\Delta K_t^H\right)$$

is now also a function of the existing allocation of capital. Only the steady state allocation of capital would coincide to the solution of Lemma 1, since $\Delta K_t^i = 0$. With adjustment costs, wealth ceases to be sufficient as the single aggregate state variable.

Unhedged incompleteness shocks To emphasize the importance of ex-ante/ex-post hedging, throughout our analysis we have assumed that agents can arrange transfers contingent on the realization of the incompleteness shocks. It is worthwhile comparing our results with the case in which productivity and incompleteness shocks cannot be hedged ex-ante. Unhedged incompleteness shocks can in principle have higher impact than hedged ones.

Production efficiency vs. constrained efficiency In our environment, it is easy to distinguish between productive efficiency, as defined in Lemma 1, and constrained-efficiency, which corresponds to a benchmark in which a planner can only intervene by submitting demands on behalf of the agents while respecting the set of available markets. From a constrained efficiency perspective, our model exclusively features distributive externalities, using the terminology of Dávila and Korinek (2017).

7 Conclusion

We have developed a parsimonious model of capital allocation in the presence of financial market distortions. We show that incompleteness shocks, defined as shocks that prevents agents from hedging future risks, even if fully anticipated and hedgeable ex-ante, have the the ability to trigger a recession with misallocation of capital, lower aggregate output and consumption. Our results highlight that the inability to share risks and to hedge ex-post is sufficient to generate a financial recession, even when agents can write ex-ante contracts that transfer resources. Our results qualify the common wisdom, expressed in Kiyotaki (2011), that both ex-ante and ex-post frictions are necessary for financial disturbances to be relevant. More broadly, if one takes as granted that a financial crisis is an event that reduces the opportunities to hedge risks in the future, any financial crisis must be associated a macroeconomic recession, independently of the ex-ante decisions made by agents.

Although we have used a canonical business cycle model to develop our results, there are many avenues for further research. In particular, we conjecture that economies in which some agents crucially rely on their ability to hedge risks, perhaps because they play an intermediary role in which having large exposures is crucial, should experience severe aggregate disruptions after a financial shock hits the economy.

Appendix

A Recursive formulation

The problem solved by agents in our model can be formulated recursively. To do so, we must specify the pertinent set of state variables. The most compact formulation adopts beginning-of-period individual wealth ω^i is the single individual state, which we define as $\omega^i = QK^i + A^i$. Experts' beginning of period wealth, W^E, as well as the current realizations of technology and incompleteness shocks form the aggregate state vector, which we denote by $\Upsilon = \{W^E, Z, \phi\}$.

The problem solved by an agent i can be expressed as

$$V^i\left(\omega^i;\Upsilon\right) = \max_{\{C^i,K^{i\prime},B^{i\prime}\}} \left[(1-\beta)\left(C^i\right)^{1-\frac{1}{\psi}} + \beta\mathbb{E}\left[V^i\left(\omega^{i\prime};\Upsilon'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{A1}$$

subject to

$$C^i \leq Z^i F\left(K^{i\prime}\right) - Q(\Upsilon) K^{i\prime} + \omega^i \tag{A2}$$

$$\omega^i = Q(\Upsilon) K^i + A^i, \tag{A3}$$

$$A^i = B^i - \phi \sum_{s'|s} q\left(s'\middle|\Upsilon\right) B^{i\prime}\left(s'\right) - (1-\phi)\kappa q^f(\Upsilon) B^{i\prime}_f, \tag{A4}$$

where $Z^i = Z$ for experts but $Z^i = 0$ for households, and where $K^{E\prime}$ and $B^{E\prime}$ are consistent with their equilibrium laws of motion.

Our formulation implies that the wealth of one the agents is sufficient as the single aggregate state variable. In this economy, aggregate net worth corresponds to the market value of the capital stock, because financial assets are in zero net supply, that is

$$\sum_i \omega^i = Q(\Upsilon)\sum_i K^i + \sum_i B^i = Q(\Upsilon)\overline{K}. \tag{A5}$$

Exploiting Equation (A5) it is possible to recover the full wealth distribution, using the fact that $W^H = Q(\Upsilon)K - W^E$. Given W^E (or W^H) it is not possible to separately recover capital and asset holdings, but this is not necessary to solve the individual problem solve by each agent. Allowing agents to trade before output is produced, the lack of adjustment costs, and the fixed supply of capital as well as the zero net supply of financial assets are crucial for this reduction in the dimensionality of the state space.

An alternative valid formulation assumes that individual capital and asset holdings are individual states, while their aggregate counterparts are the relevant aggregate states. We at times adopt this alternative formulation. A competitive equilibrium can defined in recursive form as follows.

Definition. (Recursive Competitive Equilibrium) A recursive competitive equilibrium is given by a pair of value functions $V^i\left(\omega^i;\Upsilon\right)$, $i = \{E, H\}$, consumption, capital holdings, and asset holdings for each type of agent $C^i\left(\omega^i;\Upsilon\right)$, $K^{i\prime}\left(\omega^i;\Upsilon\right)$, $B^{i\prime}\left(s'\middle|\omega_i,\Upsilon\right)$ if $\phi = 0$ or $B^{i\prime}_f\left(\omega_i,\Upsilon\right)$ if $\phi = 1$; a capital pricing function $Q(\Upsilon)$ and Arrow-Debreu or risk-free bond pricing functions $q\left(s'\middle|\Upsilon\right)$ and $q^f(\Upsilon)$; and aggregate laws of motion for experts capital and asset holdings such that: (i) given pricing functions and aggregate laws of motion, the value function and policy functions of each agent satisfy Equations (A1) to (A4), (ii) the market for capital and all financial markets clears, that is, Equations (3), (4), and (5) hold, and (iii) aggregate laws of motion are consistent with individual behavior, that is $K^{E\prime} = K^{E\prime}\left(\omega^E; \Upsilon\right)$ and $B^{E\prime} = B^{E\prime}\left(\omega^E; \Upsilon\right)$.

B Technology shocks

Any specification of technology shocks requires defining a joint stochastic process for the behavior of Z_t^E and Z_t^H. We use two different approaches.

Decoupled common and sector-specific shocks In Section 4, to highlight the difference between common and sector-specific variation, we specify Z_t^i as

$$Z_t^i = \theta_t^i Z_t,$$

which implies that we must actually specify the time series evolution of three variables: θ_t^E, θ_t^H, and Z_t. To do so, we set two targets and one normalization. First, we normalize the value of $\left(\left(\theta_t^E\right)^{\frac{1}{1-\alpha}} + \left(\theta_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} = 1$. This normalization directly implies that under production efficiency,

$$\overline{Z}_t = \left(\left(Z_t^H\right)^{\frac{1}{1-\alpha}} + \left(Z_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} = \left(\left(\theta_t^E\right)^{\frac{1}{1-\alpha}} + \left(\theta_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} Z_t = Z_t,$$

which implies that our choice of Z_t directly pins down aggregate technology. Our final target is the share of capital held by experts under complete markets. If the share is given by χ^E, we simply set $\theta_t^E = \chi^{1-\alpha}$ and $\theta_t^H = 1 - \chi^{1-\alpha}$.

Correlated common and sector-specific shocks In Section 5, to make experts "natural holders" of the risky asset, while reducing the number of technology shocks, we set $Z_t^H = 1$ and specify a univariate process for Z_t^E. In that case, under production efficiency,

$$\frac{w_t^E}{w_t^H} = \left(Z_t^E\right)^{\frac{1}{1-\alpha}}$$

$$\overline{Z}_t = \left(1 + \left(Z_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}$$

This formulation generates an endogenous correlation in levels between aggregate and idiosyncratic shocks.

C Proofs

Lemma 1 The planner's static maximization problem is given by

$$Y_t\left(\left\{Z_t^i\right\}\right) = \max_{K_t^E, K_t^H} \sum_i Z_t^i F\left(K_t^i\right),$$

subject to $\sum_i K_t^i = \overline{K}$. Where $F\left(K_t^i\right) = \left(K_t^i\right)^\alpha$ and $F'\left(K_t^i\right) = \alpha\left(K_t^i\right)^{\alpha-1}$. The solution to this problem corresponds to

$$Z_t^E F'\left(K_t^E\right) = Z_t^H F'\left(K_t^H\right) \Rightarrow \frac{K_t^E}{K_t^H} = \left(\frac{Z_t^E}{Z_t^H}\right)^{\frac{1}{1-\alpha}}$$

which combined with the resource constraint determines the solution for K_t^E and K_t^H as a function of $\overline{K}$, Z_t and α. In general, we can write

$$K_t^i = \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}} \overline{K} = w_t^{i\star}\overline{K}, \quad \text{where} \quad w_t^{i\star} = \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}}$$

and where we define $\overline{Z}_t$ as follows

$$\overline{Z}_t = \left(\sum_i \left(Z_t^i\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}$$

The aggregator $\overline{Z}_t$ can be equivalently defined as $1 = \sum_i \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}}$. Note that $\overline{Z}_t$ is a decreasing function of α. When $\alpha = 0$, the economy features almost constant returns to scale. Under symmetry, if $Z_t^i = Z_t$, then

$$\overline{Z}_t = \left(\sum_i \left(Z_t^i\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} = \left(N\left(Z_t\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} = N^{1-\alpha}Z_t$$

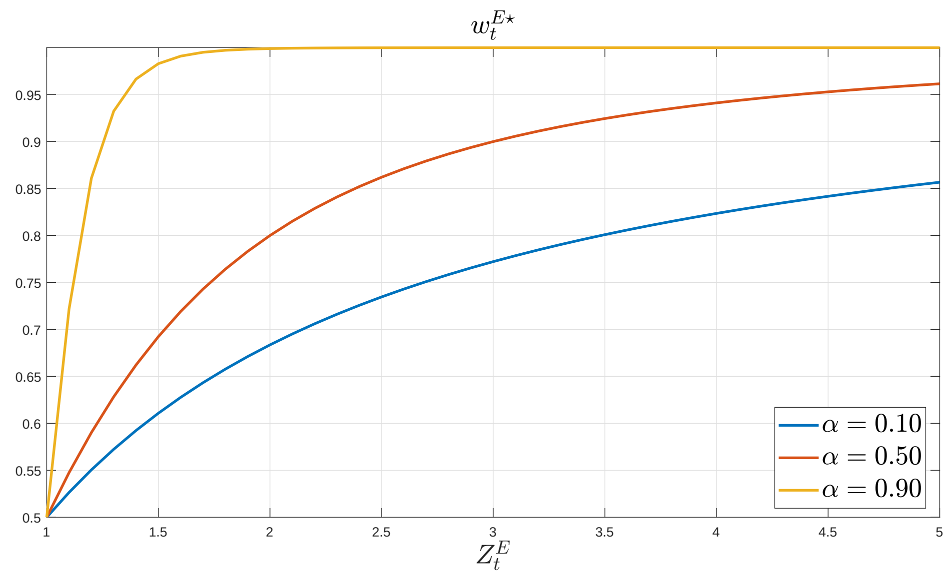


Figure A1: Efficient capital share $w_t\left(Z_t\right)$

Figure A1 illustrates how w_t varies with Z_t for different values of α.

Aggregate output is given by

$$Y_t = \max_{K_t^E, K_t^H} \sum_i Z_t^i F\left(K_t^i\right) = \sum_i Z_t^i \left(w_t^{i\star}\overline{K}\right)^{\alpha}$$

$$= \sum_i Z_t^i \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} \overline{K}^{\alpha} = \overline{Z}_t\overline{K}^{\alpha}$$

where $\overline{Z}_t$ is defined above. Note that $\frac{dY_t}{d\alpha}$ is ambiguous. For low value of $\overline{K}$ it's better to have a low α since decreasing returns have not kicked in yet. For high values of $\bar{K}$ it's better to have high values of α, since it's more efficient to allocate more capital to the best agents and make use of CRS.

Note also that

$$Y_t^i = Z_t^i F\left(K_t^i\right) = Z_t^i \left(\left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}} \overline{K}\right)^{\alpha} = \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}} \overline{Z}_t \overline{K}^{\alpha} = w_t^{i\star} \overline{Z}_t \overline{K}^{\alpha}$$

Agents' optimization problem Both experts and households solve

$$V^i\left(\omega^i; Y\right) = \max_{\left\{C^i, K^{i\prime}, B^{i\prime}\right\}} \left[(1-\beta)\left(C^i\right)^{1-\frac{1}{\psi}} + \beta \mathbb{E}\left[V^i\left(\omega^{i\prime}; Y^{\prime}\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}},$$

subject to

$$C^i \leq Z^i F\left(K^{i\prime}\right) - Q\left(Y\right) \Delta K^{i\prime} + B^i - \phi \sum_{s^{\prime}|s} q\left(s^{\prime}\middle| Y\right) B^{i\prime}\left(s^{\prime}\right) - (1-\phi)\kappa q^f\left(Y\right) B_f^{i\prime}$$

The marginal utility of consumption at period t, which we denote by λ_t^E, is given by $\lambda_t^E = \left(V_t^E\right)^{\frac{1}{\psi}} \left(C_t^E\right)^{-\frac{1}{\psi}}$. The optimality condition for capital implies that

$$\lambda_t^E\left(Q_t - Z_t F^{\prime}\left(K_t^E\right)\right) = \beta\left(V_t^E\right)^{\frac{1}{\psi}} \mathbb{E}_t\left[\left(V_{t+1}^E\right)^{1-\gamma}\right]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \mathbb{E}_t\left[\left(V_{t+1}^E\right)^{-\gamma} \frac{\partial V_{t+1}^E}{\partial K_t^E}\right],$$

which combined with the envelope theorem, $\frac{\partial V_{t+1}^E}{\partial K_t^E} = \lambda_{t+1}^E Q_{t+1}$ and $\frac{\partial V_{t+1}^E}{\partial B_{t+1}^E} = \lambda_{t+1}^E$, and defining $\Upsilon_t \equiv \mathbb{E}_t\left[\left(V_{t+1}^E\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$ yields

$$Q_t = Z_t F^{\prime}\left(K_t^E\right) + \mathbb{E}_t\left[\beta\left(\frac{V_{t+1}^E}{\Upsilon_t}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}^E}{C_t^E}\right)^{-\frac{1}{\psi}} Q_{t+1}\right]$$

Where we define $m_{t+1}^E = \beta\left(\frac{V_{t+1}^E}{\zeta_t^E}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}^E}{C_t^E}\right)^{-\frac{1}{\psi}}$, as in Equation (8) in the text. When $\phi_t = 1$, the optimality conditions for the Arrow-Debreu securities is given by

$$q_t\left(s^{t+1}\middle| s^t\right) = \pi\left(s^{t+1}\middle| s^t\right)\beta\left(\frac{V_{t+1}^E}{\Upsilon_t}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}^E}{C_t^E}\right)^{-\frac{1}{\psi}}.$$

So $q_t\left(s^{t+1}\middle| s^t\right) = \pi\left(s^{t+1}\middle| s^t\right) m_{t+1}^E\left(s^{t+1}\middle| s^t\right)$. When $\phi_t = 0$ and $\kappa = 1$, the optimality condition for the riskless bond is given by

$$q_t^f = \mathbb{E}_t\left[\beta\left(\frac{V_{t+1}^E}{\Upsilon_t}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{C_{t+1}^E}{C_t^E}\right)^{-\frac{1}{\psi}}\right].$$

Lemma 2 Equations (9) and (11), which hold under complete markets, when combined imply the condition for production efficiency

$$Z_t^E F^{\prime}\left(K_t^E\right) - Z_t^H F^{\prime}\left(K_t^H\right) = 0.$$

Proposition 1 Follows from Equation (11), as explained in the text.

Proposition 2 Combining the optimality conditions from both experts and households, we can write

$$\Omega_t = Z_t^E F'\left(K_t^E\right) - Z_t^H F'\left(K_t^H\right),$$

where we define

$$\Omega_t \equiv \mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right].$$

We can also write

$$\tilde{\Omega}_t = 1 - \frac{Z_t^H F'\left(K_t^H\right)}{Z_t^E F'\left(K_t^E\right)},$$

where we define

$$\tilde{\Omega}_t \equiv \frac{\mathbb{E}_t\left[m_{t+1}^H Q_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^E Q_{t+1}\right]}{Z_t^E F'\left(K_t^E\right)}.$$

Note that $\frac{Z_t^E}{Z_t^H}\left(1 - \tilde{\Omega}_t\right) = \frac{F'\left(K_t^H\right)}{F'\left(K_t^E\right)}$, which implies that $\frac{K_t^E}{K_t^H}$ is a decreasing function of Ω_t and allows us to write

$$\frac{K_t^E}{K_t^H} = \left(\frac{Z_t^E\left(1-\tau_t\right)}{Z_t^H\left(1+\tau_t\right)}\right)^{\frac{1}{1-\alpha}},$$

where we define τ_t as the solution to

$$1 - \tilde{\Omega}_t = \frac{1-\tau_t}{1+\tau_t}.$$

We can therefore write

$$K_t^i = w_t^i \overline{K},$$

where

$$w_t^i = \left(\frac{\tilde{Z}_t^i}{\overline{\tilde{Z}}_t}\right)^{\frac{1}{1-\alpha}},$$

and we define

$$\tilde{Z}_t^E = Z_t^E\left(1-\tau_t\right)$$

$$\tilde{Z}_t^H = Z_t^H\left(1+\tau_t\right)$$

and

$$\overline{\tilde{Z}}_t == \left(\left(\tilde{Z}_t^H\right)^{\frac{1}{1-\alpha}} + \left(\tilde{Z}_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}.$$

Note that we can relate $\overline{Z}_t$ and $\overline{\tilde{Z}}_t$ as follows

$$\frac{\overline{Z}_t}{\overline{\tilde{Z}}_t} = \frac{\left(\left(Z_t^H\right)^{\frac{1}{1-\alpha}} + \left(Z_t^E\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}}{\left(\left(Z_t^H\left(1+\tau_t\right)\right)^{\frac{1}{1-\alpha}} + \left(Z_t^E\left(1-\tau_t\right)\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}}.$$

We can express individual production as

$$
\begin{aligned}
Z_t^i F\left(K_t^i\right) &= Z_t^i \left(\left(\frac{\tilde{Z}_t^i}{\overline{\tilde{Z}}_t}\right)^{\frac{1}{1-\alpha}} \overline{K}\right)^{\alpha} = \frac{Z_t^i}{\overline{Z}_t}\left(\frac{\tilde{Z}_t^i}{\overline{\tilde{Z}}_t}\right)^{\frac{\alpha}{1-\alpha}} \overline{Z}_t \overline{K}^{\alpha} \\
&= \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{1-\frac{1}{1-\alpha}} \left(\frac{\tilde{Z}_t^i}{\overline{\tilde{Z}}_t}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{Z_t^i}{\overline{Z}_t}\right)^{\frac{1}{1-\alpha}} \overline{Z}_t \overline{K}^{\alpha} \\
&= \left(\frac{\tilde{Z}_t^i/\overline{\tilde{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} w_t^{i\star} \overline{Z}_t \overline{K}^{\alpha} \\
&= \zeta_t^i w_t^{i\star} \overline{Z}_t \overline{K}^{\alpha}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Y_t &= Z_t^E F\left(K_t^E\right) + Z_t^H F\left(K_t^E\right) \\
&= \sum_i \left(\frac{\tilde{Z}_t^i/\overline{\tilde{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} w_t^{i\star} \overline{Z}_t \overline{K}^{\alpha}
\end{aligned}
$$

Given that $w_t^{i\star}$ are production maximizing weights, it must be that

$$
\sum_i \left(\frac{\tilde{Z}_t^i/\overline{\tilde{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} w_t^i \leq 1,
$$

with equality if $Z_t^i = \tilde{Z}_t^i$. Since $\sum_i w_t^{i\star} = 1$. It must be the case then that $\sum_i \left(\frac{\tilde{Z}_t^i/\overline{\tilde{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} \leq 1$.

Consistently with the definition in Proposition 2, we can define a TFP wedge as follows:

$$
\Xi\left(\Omega_t\right) = \frac{Y_t}{\overline{Z}_t \overline{K}^{\alpha}}
$$

where $\Xi\left(\Omega_t\right) = \sum_i \left(\frac{\tilde{Z}_t^i/\overline{\tilde{Z}}_t}{Z_t^i/\overline{Z}_t}\right)^{\frac{\alpha}{1-\alpha}} w_t^{i\star}$ is a decreasing function of $|\Omega_t|$.

Proposition 3 This result follows from the optimality conditions. The necessary condition for production efficiency is that expert's and household's continuation value for the risky asset are identical. Whenever market are incomplete in period t, generically for any set of state values that condition will not be satisfied.

Proposition 4 (to be added) Proof by contradiction. Assume that the first best allocation is an equilibrium. This is inconsistent with the conditions for optimality.

Proposition 5 (to be added) Proof by contradiction.

References

Acemoglu, D., and F. Zilibotti. 1997. "Was Prometheus unbound by chance? Risk, diversification, and growth." *Journal of political economy*, 105(4): 709–751.

Angeletos, George-Marios. 2002. "Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure." *The Quarterly Journal of Economics*, 117(3): 1105–1131.

Bernanke, B., and M. Gertler. 1989. "Agency Costs, Net Worth, and Business Fluctuations." *The American Economic Review*, 14–31.

Bernanke, Ben S, Mark Gertler, and Simon Gilchrist. 1999. "The financial accelerator in a quantitative business cycle framework." *Handbook of macroeconomics*, 1: 1341–1393.

Brunnermeier, Markus K, and Yuliy Sannikov. 2014. "A macroeconomic model with a financial sector." *The American Economic Review*, 104(2): 379–421.

Buera, Francisco, and Juan Pablo Nicolini. 2004. "Optimal maturity of government debt without state contingent bonds." *Journal of Monetary Economics*, 51(3): 531–554.

Buera, Francisco, and Juan Pablo Nicolini. 2015. "Liquidity traps and monetary policy: Managing a credit crunch."

Buera, Francisco J, and Benjamin Moll. 2014. "Aggregate Implications of a Credit Crunch." *Working Paper*.

Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian. 2016. "Optimal Contracts, Aggregate Risk, and the Financial Accelerator." *American Economic Journal: Macroeconomics*, 8(1): 119–47.

Carlstrom, C.T., and T.S. Fuerst. 1997. "Agency costs, net worth, and business fluctuations: A computable general equilibrium analysis." *The American Economic Review*, 893–910.

Christiano, Lawrence J, Roberto Motto, and Massimo Rostagno. 2014. "Risk Shocks." *The American Economic Review*, 104(1): 27–65.

Cole, Harold. 2011. "Discussion of Gertler and Karadi: A model of unconventional monetary policy." *Journal of Monetary Economics*, 58(1): 35–38.

Coleman, Wilbur John. 1990. "Solving the stochastic growth model by policy-function iteration." *Journal of Business & Economic Statistics*, 8(1): 27–29.

Coleman, Wilbur John. 1991. "Equilibrium in a production economy with an income tax." *Econometrica: Journal of the Econometric Society*, 1091–1104.

Cooley, Thomas, Ramon Marimon, and Vincenzo Quadrini. 2004. "Aggregate consequences of limited contract enforceability." *Journal of Political Economy*, 112(4): 817–847.

Cordoba, Juan-Carlos, and Marla Ripoll. 2004. "Credit Cycles Redux." *International Economic Review*, 45(4): 1011–1046.

David, Joel M, Hugo A Hopenhayn, and Venky Venkateswaran. 2016. "Information, misallocation and aggregate productivity." *The Quarterly Journal of Economics*, qjw006.

Dávila, Eduardo, and Anton Korinek. 2017. "Pecuniary Externalities in Economies with Financial Frictions." *The Review of Economic Studies*, Forthcoming.

Di Tella, Sebastian. 2017. "Uncertainty Shocks and Balance Sheet Recessions." *Journal of Political Economy, Forthcoming*.

Eberly, Janice, and Neng Wang. 2009. "Capital reallocation and growth." *The American Economic Review*, 99(2): 560–566.

Eberly, Janice C, and Neng Wang. 2011. "Reallocating and pricing illiquid capital: Two productive trees." *Working Paper*.

Greenwood, J., and B. Jovanovic. 1990. "Financial Development, Growth, and the Distribution of Income." *The Journal of Political Economy*, 98(5 Part 1): 1076–1107.

He, Z., and A. Krishnamurthy. 2012. "A model of capital and crises." *The Review of Economic Studies*, 79(2): 735–777.

Jeanne, Oliver, and Anton Korinek. 2010. "Managing credit booms and busts: A Pigouvian taxation approach." *Working paper*.

Jermann, Urban, and Vincenzo Quadrini. 2012. "Macroeconomic Effects of Financial Shocks." *American Economic Review*, 102(1): 238–71.

Judd, K.L. 1998. *Numerical methods in economics.* The MIT press.

Kiyotaki, N., and J. Moore. 1997. "Credit Cycles." *The Journal of Political Economy*, 105(2): 211–248.

Kiyotaki, Nobuhiro. 2011. "A Perspective on Modern Business Cycle Theory." *Economic Quarterly, Federal Reserve Bank of Richmond*, 97(3): 195.

Kocherlakota, Narayana R. 2000. "Creating business cycles through credit constraints." *Federal Reserve Bank of Minneapolis. Quarterly Review-Federal Reserve Bank of Minneapolis*, 24(3): 2.

Kogan, Leonid. 2001. "An equilibrium model of irreversible investment." *Journal of Financial Economics*, 62(2): 201–245.

Kogan, Leonid. 2004. "Asset prices and real investment." *Journal of Financial Economics*, 73(3): 411–431.

Krishnamurthy, Arvind. 2003. "Collateral constraints and the amplification mechanism." *Journal of Economic Theory*, 111(2): 277–292.

Li, Huiyu, and John Stachurski. 2014. "Solving the income fluctuation problem with unbounded rewards." *Journal of Economic Dynamics and Control*, 45: 353–365.

Ljungqvist, L., and T.J. Sargent. 2004. *Recursive macroeconomic theory.* The MIT press.

Lustig, Hanno, Christian Sleet, and Sevin Yeltekin. 2008. "Fiscal hedging with nominal assets." *Journal of Monetary Economics*, 55(4): 710–727.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic theory.* Oxford University Press.

Mendoza, Enrique G. 2010. "Sudden stops, financial crises, and leverage." *American Economic Review*, 100(5): 1941–1966.

Moll, Benjamin. 2014. "Productivity losses from financial frictions: can self-financing undo capital misallocation?" *The American Economic Review*, 104(10): 3186–3221.

Perri, Fabrizio, and Vincenzo Quadrini. 2017. "International Recessions." *American Economic Review (Forthcoming)*.

Rampini, A., and S. Viswanathan. 2010. "Collateral, risk management, and the distribution of debt capacity." *The Journal of Finance*, 65(6): 2293–2322.

Rendahl, Pontus. 2015. "Inequality Constraints and Euler Equation-based Solution Methods." *The Economic Journal*, 125(585): 1110–1135.

ONLINE APPENDIX

We next describe the algorithm that corresponds to the numerical solution of the model. We solve the model globally using a time iteration procedure, which at times is referred to as policy function iteration or as an Euler equation-base solution method. See Chapter 17 of Judd (1998) or Rendahl (2015) for details on this class of solution methods.

D Numerical Solution

We sequentially describe the five models that we solve in this paper. The first four are benchmarks. These are: i) Autarky, ii) Incomplete markets model in which only capital is traded, iii) Incomplete markets model in which capital and a single non-contingent bond are traded, and iv) Complete markets model in which capital and all Arrow-Debreu securities are traded. The fifth and most general model is the main object of study in this paper. Table A1 describes the different benchmarks, highlighting the set of minimum relevant state variables and the outcomes of interest. Note that in our simulations, we often adopt K^E and B^E as state variables instead of W^E, to improve the numerical stability of the solution.

Table A1: Benchmarks

Model	Individual States	Aggregate States Y	Outcomes
AUTARKY	n/a	K^E, Z	$V^i, C^i, K^{i\prime}, Q$
ONLY CAPITAL	k^i (or ω^i)	W^E, Z	$V^i, C^i, K^{i\prime}, Q$
CAPITAL + BOND	ω^i	W^E, Z	$V^i, C^i, K^{i\prime}, B^i, Q, q_s$
COMPLETE MARKETS	ω^i	W^E, Z	$V^i, C^i, K^{i\prime}, B^i_s, Q, q_s$
INCOMPLETENESS SHOCKS	ω^i	W^E, Z, ϕ	$V^i, C^i, K^{i\prime}, B^i_s, Q, q_s, q_f$

D.1 Autarky

We initially solve a benchmark model without capital or financial markets. Assuming that agents start with a given allocation of capital that satisfies the capital resource constraint, and denoting the current aggregate state as $Y = (K^E, Z)$, a solution of the model consists of finding value functions $V^i(Y)$, consumption functions $C^i(Y)$, as well as shadow prices $Q^i(Y)$ for each agent i such that

$$V^i(Y) = \left[(1-\beta)\left(C^i(Y)\right)^{1-\frac{1}{\psi}} + \beta \mathbb{E}\left[V^i\left(K^E, Z'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}},$$

where the consumption functions are trivially given by $C^E(Y) = ZF(K^E)$ and $C^H(Y) = F(\overline{K} - K^E)$. By virtue of the autarky assumptions, capital holdings remain constant.

The entrepreneur's (shadow) price of capital $Q^E(Y)$ satisfies:

$$\left[Q^E(Y) - ZF'\left(K^{E\prime}\right)\right]\left(C^E(Y)\right)^{-\frac{1}{\psi}} = \beta \mathbb{E}\left[\left(\frac{V^E(K^E, Z')}{\zeta^E(Y)}\right)^{\frac{1}{\psi}-\gamma}\left(C^E\left(K^E, Z'\right)\right)^{-\frac{1}{\psi}} Q\left(K^E, Z'\right)\right],$$

$$\text{where} \quad \zeta^E(Y) = \mathbb{E}\left[V^E\left(K^E, Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}.$$

Because households face no uncertainty, their shadow price of capital $Q^H(K^E, Z) = Q^H(K^E)$ is trivially given by

$$Q^H\left(K^E, Z\right) = \frac{F'\left(\overline{K} - K^E\right)}{1-\beta},$$

which is independent of Z.

One can plug in for the trivial form of $C^i(Y)$ into the Bellman equation and the determination of the entrepreneur's shadow price of capital, to collapse the problem of finding a solution into determining the functions $\left\{V^i(Y), Q^i(Y)\right\}_{i\in\{E,H\}}$

that satisfy these conditions. The following subsection discusses in more detail the general method of how we solve all the models.

To be brief, we discretize the continuous state space $\left[0, \overline{K}\right]$ with n_K values and solve for the equilibrium objects on a finite number of grid points $(K_l, Z_m) \in \left[0, \overline{K}\right] \times \mathcal{Z}$ where $K_l \in \{K_l\}_{l \in \{1,\ldots n_K\}} \subset \left[0, \overline{K}\right]$, $Y = (K, Z) \in \left[0, \overline{K}\right] \times \mathcal{Z}$, with $\mathcal{Z}$ being the set of n_Z potential realizations for TFP Z.

We adopt a time iteration procedure (see next subsection), by making initial guesses for the values of $\left\{V^i(Y), Q^i(Y)\right\}_{i \in \{E,H\}}$ for all $Y \in \{K_l\}_{l \in \{1,\ldots n_K\}} \times \mathcal{Z}$ on the discrete grid, resulting in $n_K \times n_Z$ matrices $\left\{\hat{M}_0^{Vi}, \hat{M}_0^{Qi}\right\}_{i \in \{E,H\}}$ with the (l,m)-entry of the matrix $\hat{M}_0^j$ referring to the initial guess of equilibrium function f^j evaluated at $(K_l, Z_m) \in \left[0, \overline{K}\right] \times \mathcal{Z}$. Given $\left\{\hat{M}_\tau^{Vi}, \hat{M}_\tau^{Qi}\right\}_{i \in \{E,H\}}$ and the equilibrium object values on the grid that they imply $\left\{\hat{V}^i(Y), \hat{Q}^i(Y)\right\}_{i \in \{E,H\}}$, at any iteration $\tau + 1$, for every $(K_l, Z_m) \in \left[0, \overline{K}\right] \times \mathcal{Z}$ we then solve for the $\left\{q_{lm}^E, q_{lm}^H, v_{lm}^E, v_{lm}^H\right\}$ that satisfy:

$$\left[q_{lm}^E - Z_m F'(K_l)\right] \left(Z_m F(K_l)\right)^{-\frac{1}{\psi}} = \beta \mathbb{E}\left[\left(\frac{\hat{V}_\tau^E(K_l, Z')}{\hat{\zeta}_\tau^E(K_l, Z)}\right)^{\frac{1}{\psi} - \gamma} \left(Z' F(K_l)\right)^{-\frac{1}{\psi}} \hat{Q}_\tau^E(K_l, Z')\right]$$

$$q_{lm}^H = \frac{F'\left(\overline{K} - K_l\right)}{1 - \beta}$$

$$v_{lm}^E = \left[(1-\beta)\left(ZF(K_l)\right)^{1-\frac{1}{\psi}} + \beta \hat{\zeta}_\tau^E(K_l, Z_m)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

$$v_{lm}^H = \left[(1-\beta)\left(F\left(\overline{K} - K_l\right)\right)^{1-\frac{1}{\psi}} + \beta \hat{\zeta}_\tau^H(K_l, Z_m)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

$$\text{with } \hat{\zeta}_\tau^i(K_l, Z_m) = \mathbb{E}\left[\hat{V}_\tau^i(K_l, Z')^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, \text{ for } i \in \{E, H\}$$

And then set

$$\left\{\hat{M}_{\tau+1}^{QE}(l,m), \hat{M}_{\tau+1}^{QH}(l,m), \hat{M}_{\tau+1}^{VE}(l,m), \hat{M}_{\tau+1}^{VH}(l,m)\right\} = \left\{q_{lm}^E, q_{lm}^H, v_{lm}^E, v_{lm}^H\right\},$$

with $\hat{M}_{\tau+1}^j(l,m)$ referring to the row-l, column-m entry of $\hat{M}_{\tau+1}^j$.

We iterate until $\max_{l,m,j}\left|\hat{M}_\tau^j(l,m) - \hat{M}_\tau^j(l,m)\right| \leq \varepsilon$, for some small ε.

D.2 Capital Only

We provide the most detailed description of the computational algorithm we employ for the model in which only capital is traded. The model is more easily formulated with $Y = \left(K^E, Z\right)$ as aggregate state variables. In this case agent $i \in \{E, H\}$ solves the problem

$$V^i\left(\omega^i; Y\right) = \max_{\{C^i, K^{i\prime}, B^{i\prime}\}} \left[(1-\beta)\left(C^i\right)^{1-\frac{1}{\psi}} + \beta \mathbb{E}\left[V^i\left(\omega^{i\prime}; Y'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{A6}$$

subject to

$$C^i \leq Z^i F\left(K^{i\prime}\right) - Q(Y) K^{i\prime} + \omega^i \tag{A7}$$

$$\omega^i = Q(Y) K^i. \tag{A8}$$

The FOC for $K^{i\prime}$ can be written, as seen in the main text, as

$$\left[Q(Y) - Z^i F'\left(K^{i\prime}\right)\right]\left(C^i\right)^{-\frac{1}{\psi}} = \beta \mathbb{E}\left[\left(\frac{V^i\left(Q(Y') K^{i\prime}; Y'\right)}{\zeta^{i\prime}}\right)^{\frac{1}{\psi} - \gamma}\left(C^{i\prime}\right)^{-\frac{1}{\psi}} Q(Y')\right],$$

$$\text{where} \quad \zeta^{i\prime} = \mathbb{E}\left[V^i\left(Q(Y') K^{i\prime}; Y'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

This FOC, together with (A6)–(A8) and the corresponding laws of motion determine agent i's individual optimal choices of $C^i\left(\omega^i; Y\right)$, $K^{i\prime}\left(\omega^i; Y\right)$ and the value of $V^i\left(\omega^i; Y\right)$ for any given $\left(\omega^i; Y\right)$.

In equilibrium, in any aggregate state $Y = \left(K^E, Z\right)$, it must be the case that the aggregate law of motion of the endogenous state $K^{E\prime}(\mathrm{Y})$ and the price of capital $Q(\mathrm{Y})$ is consistent with the optimality of both agents $i \in \{E, H\}$ and the capital market clearing condition. And of course, in any aggregate state Y, both agents' individual wealth is consistent with the state: $\omega^E = Q(\mathrm{Y}) K^E$, $\omega^H = Q(\mathrm{Y})\left(\overline{K} - K^E\right)$. Once we have derived both agents' optimality conditions, we can drop the dependence of individuals' choices on ω^i, as the aggregate state already determines both agents' wealth levels (i.e. we are equalizing "big-K" and "little-k"), and think of the equilibrium in this "Capital Only" model being defined as follows.

Definition. (Recursive equilibrium in "Capital Only" model) A recursive competitive equilibrium in the "Capital Only" model is given by a pair of value functions $V^i(Y)$, $i = \{E, H\}$, consumption and capital holdings policy functions for each type of agent $C^i(Y)$ and $K^{i\prime}(Y)$; and a capital pricing function $Q(Y)$ such that the following equilibrium conditions are satisfied for any $\mathrm{Y} = \left(K^E, Z\right)$:

$$\left[Q(\mathrm{Y}) - Z^i F'\left(K^{i\prime}(\mathrm{Y})\right)\right]\left(C^i(\mathrm{Y})\right)^{-\frac{1}{\psi}} = \beta \mathbb{E}\left[\left(\frac{V^i\left(K^{E\prime}(\mathrm{Y}), Z'\right)}{\zeta^i\left(K^{E\prime}(\mathrm{Y}), Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(C^{i\prime}\left(K^{E\prime}(\mathrm{Y}), Z'\right)\right)^{-\frac{1}{\psi}} Q\left(K^{E\prime}(\mathrm{Y}), Z'\right)\right], \text{ for } i \in \{E, H\}$$

$$\text{where} \quad \zeta^i\left(K^{E\prime}(\mathrm{Y}), Z\right) = \mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$C^i(\mathrm{Y}) = Z^i F\left(K^{i\prime}(\mathrm{Y})\right) - Q(\mathrm{Y})\left[K^{i\prime}(\mathrm{Y}) - K^i\right], \text{ for } i \in \{E, H\}$$
$$\text{with } Z^E = Z,\ Z^H = 1 \text{ and } K^H = \overline{K} - K^E$$

$$V^i(\mathrm{Y}) = \left[(1-\beta)\left(C^i(\mathrm{Y})\right)^{1-\frac{1}{\psi}} + \beta \mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \text{ for } i \in \{E, H\}$$

$$K^{E\prime}(\mathrm{Y}) + K^{H\prime}(\mathrm{Y}) = \overline{K}$$

For any specific current aggregate state Y_v, one could think of this as a system of 7 equations pinning down 7 numbers: $\left\{C^i(\mathrm{Y}_v), K^{i\prime}(\mathrm{Y}_v), V^i(\mathrm{Y}_v)\right\}_{i\in\{E,H\}}$ and $Q(\mathrm{Y}_v)$. And solving for an equilibrium amounts to determining the functions $f \equiv \left\{\left\{C^i(\mathrm{Y}), K^{i\prime}(\mathrm{Y}), V^i(\mathrm{Y})\right\}_{i\in\{E,H\}}, Q(\mathrm{Y})\right\}$ that satisfy the above system for any $\mathrm{Y} = (K, Z) \in \left[0, \overline{K}\right] \times \mathcal{Z}$, with $\mathcal{Z}$ being the set of n_Z potential realizations for TFP Z.

To make the task of solving for these functions f feasible on a computer, we can approximate them using finite approximations $\hat{f}$ chosen from some restricted space of functions defined on $\left[0, \overline{K}\right] \times \mathcal{Z}$. The approximation is then determined by solving the equilibrium conditions on a finite number of grid points $(K_l, Z) \in \left[0, \overline{K}\right] \times \mathcal{Z}$ where $K_l \in \{K_l\}_{l\in\{1,\dots n_K\}} \subset \left[0, \overline{K}\right]$ is an element in the discretization of the continuous state space dimension $\left[0, \overline{K}\right]$ into n_K points. Whenever $\hat{f}$ must be evaluated outside of the grid points, interpolation of values on the grid points is used. We use throughout piecewise linear functions defined on the continuous, discretized state space components, i.e. multilinear interpolation, for each realization on the discrete space of Z. That is, in the "Capital Only" model, an equilibrium function f^j is approximated by a collection of n_Z piecewise linear functions $\left\{\hat{f}^j_{Z_m}\right\}_{Z_m\in\mathcal{Z}}$, with $\hat{f}^j_{Z_m}(K) \approx f^j(K, Z_m)$, and each $\hat{f}^j_{Z_m}$ being defined by a vector of length n_K – its values on the discretized points in K-space, given $Z = Z_m$. Or, equivalently, the approximant $\hat{f}^j$ is defined by a $n_K \times n_Z$ matrix $\hat{M}^j$ of values, with the (l, m)-entry corresponding to $\hat{f}^j_{Z_m}(K_l)$. And the computational task amounts to finding $n_K \times n_Z$ matrices, one for each function in f, such that the equilibrium conditions are satisfied by the implied approximants $\hat{f}$ at each point $(K_{l,}, Z_m) \in \{K_l\}_{l\in\{1,\dots n_K\}} \times \mathcal{Z}$.

Before we go to the detailed setup of the computational algorithm of solving for $\hat{f}$ (effectively $\hat{M}$), note that the general equilibrium conditions can be combined further and condensed into a system of four equations in only $K^{E\prime}(\mathrm{Y})$, $V^E(\mathrm{Y})$, $V^H(\mathrm{Y})$ and $Q(\mathrm{Y})$. These are

$$\left[Q(\mathrm{Y}) - ZF'\left(K^{E\prime}(\mathrm{Y})\right)\right]\left(ZF\left(K^{E\prime}(\mathrm{Y})\right) - Q(\mathrm{Y})\left[K^{E\prime}(\mathrm{Y}) - K^E\right]\right)^{-\frac{1}{\psi}} =$$
$$\beta \mathbb{E}\left[\left(\frac{V^E\left(K^{E\prime}(\mathrm{Y}), Z'\right)}{\zeta^E\left(K^{E\prime}(\mathrm{Y}), Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(Z'F\left(K^{E\prime}\left(K^{E\prime}(\mathrm{Y}), Z'\right)\right) - Q\left(K^{E\prime}(\mathrm{Y}), Z'\right)\left[K^{E\prime}\left(K^{E\prime}(\mathrm{Y}), Z'\right) - K^{E\prime}(\mathrm{Y})\right]\right)^{-\frac{1}{\psi}} Q\left(K^{E\prime}(\mathrm{Y}), Z'\right)\right]$$

$$\left[Q\left(\mathrm{Y}\right)-F'\left(\overline{K}-K^{E\prime}\left(\mathrm{Y}\right)\right)\right]\left(F\left(\overline{K}-K^{E\prime}\left(\mathrm{Y}\right)\right)-Q\left(\mathrm{Y}\right)\left[K^{E}-K^{E\prime}\left(\mathrm{Y}\right)\right]\right)^{-\frac{1}{\psi}}=$$

$$\beta\mathbb{E}\left[\left(\frac{V^{H}\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)}{\zeta^{H}\left(K^{E\prime}\left(\mathrm{Y}\right),Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(F\left(\overline{K}-K^{E\prime}\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)\right)-Q\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)\left[K^{E\prime}-K^{E\prime}\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)\right]\right)^{-\frac{1}{\psi}}Q\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)\right]$$

$$V^{E}\left(\mathrm{Y}\right)=\left[\left(1-\beta\right)\left(ZF\left(K^{E\prime}\left(\mathrm{Y}\right)\right)-Q\left(\mathrm{Y}\right)\left[K^{E\prime}\left(\mathrm{Y}\right)-K^{E}\right]\right)^{1-\frac{1}{\psi}}+\beta\zeta^{E}\left(K^{E\prime},Z\right)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

$$V^{H}\left(\mathrm{Y}\right)=\left[\left(1-\beta\right)\left(F\left(\overline{K}-K^{E\prime}\left(\mathrm{Y}\right)\right)-Q\left(\mathrm{Y}\right)\left[K^{E}-K^{E\prime}\left(\mathrm{Y}\right)\right]\right)^{1-\frac{1}{\psi}}+\beta\zeta^{H}\left(K^{E\prime},Z\right)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

$$\text{where}\quad \zeta^{i}\left(K^{E\prime}\left(\mathrm{Y}\right),Z\right)=\mathbb{E}\left[V^{i}\left(K^{E\prime}\left(\mathrm{Y}\right),Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}},\ \text{for } i\in\{E,H\}$$

To solve for the approximations $\hat{f}=\left\{\hat{K}^{E\prime}\left(\mathrm{Y}\right),\hat{V}^{E}\left(\mathrm{Y}\right),\hat{V}^{H}\left(\mathrm{Y}\right),\hat{Q}\left(\mathrm{Y}\right)\right\}$ (again, equivalently, the matrices $\hat{M}=\{\hat{M}^{K},\hat{M}^{VE},\hat{M}^{VH},\hat{M}^{Q}\}$), we employ time iteration (Judd (1998), p. 553). That is, for any given interpolators $\hat{f}_{\tau}=\left\{\hat{K}^{E\prime}_{\tau}\left(\mathrm{Y}\right),\hat{V}^{E}_{\tau}\left(\mathrm{Y}\right),\hat{V}^{H}_{\tau}\left(\mathrm{Y}\right),\hat{Q}_{\tau}\left(\mathrm{Y}\right)\right\}$, again, encompassed in the matrices $\hat{M}_{\tau}=\left\{\hat{M}^{K}_{\tau},\hat{M}^{VE}_{\tau},\hat{M}^{VH}_{\tau},\hat{M}^{Q}_{\tau}\right\}$, we impose that equilibrium variables in period $t+1$ are as determined by $\hat{f}_{\tau}$, and solve for equilibrium values in t that satisfy the equilibrium conditions at every grid point in $\left(K_{l},Z_{m}\right)\in\{K_{l}\}_{l\in\{1,\ldots n_{K}\}}\times\mathcal{Z}$, determining $\hat{M}_{\tau+1}$. And we then iterate until $\hat{M}_{\tau+1}$ and $\hat{M}_{\tau}$ are sufficiently close based on some convergence criterion. This method has proven fast, widely applicable, and accurate in practice, despite the fact that its convergence properties are less well studied than, for instance, value function iteration methods. Coleman (1990, 1991) contains the seminal application of this methodology. Li and Stachurski (2014) and Rendahl (2015) provide convergence results. Judd (1998) provides a textbook exposition of the methodology. The detailed algorithm then looks as follows.

1. Choose an initial guess for $\hat{M}_{0}$ – a collection of four $n_{K}\times n_{Z}$ matrices. As a simple initial guess, we propose $\hat{M}^{K}_{0}$ and $\hat{M}^{Q}_{0}$ such that for any current Z the implied $K^{E\prime}$ and Q are at the non-stochastic steady state values corresponding to the unconditional mean of Z. We set both $\hat{M}^{VE}_{0}$ and $\hat{M}^{VH}_{0}$ to just constants at the value 1. Set $\hat{M}_{\tau}=\hat{M}_{0}$ and start the following iteration:
   (a) Given the approximating functions $\hat{f}_{\tau}$ implied by $\hat{M}_{\tau}$, at every $\left(K_{l},Z_{m}\right)\in\{K_{l}\}_{l\in\{1,\ldots n_{K}\}}\times\mathcal{Z}$ solve the following system for $\{k_{lm},v^{E}_{lm},v^{H}_{lm},q_{lm}\}$:

$$\left[q_{lm}-Z_{m}F'\left(k_{lm}\right)\right]\left(Z_{m}F\left(k_{lm}\right)-q_{lm}\left[k_{lm}-K_{l}\right]\right)^{-\frac{1}{\psi}}$$
$$=\beta\mathbb{E}\left[\left(\frac{\hat{V}^{E}_{\tau}\left(k_{lm},Z'\right)}{\hat{\zeta}^{E}_{\tau}\left(k_{lm},Z_{m}\right)}\right)^{\frac{1}{\psi}-\gamma}\left(Z'F\left(\hat{K}^{E\prime}_{\tau}\left(k_{lm},Z'\right)\right)-\hat{Q}_{\tau}\left(k_{lm},Z'\right)\left[\hat{K}^{E\prime}_{\tau}\left(k_{lm},Z'\right)-k_{lm}\right]\right)^{-\frac{1}{\psi}}\hat{Q}_{\tau}\left(k_{lm},Z'\right)\right]$$

$$\left[q_{lm}-F'\left(k_{lm}\right)\right]\left(F\left(\overline{K}-k_{lm}\right)-q_{lm}\left[K_{l}-k_{lm}\right]\right)^{-\frac{1}{\psi}}$$
$$=\beta\mathbb{E}\left[\left(\frac{\hat{V}^{H}_{\tau}\left(k_{lm},Z'\right)}{\hat{\zeta}^{H}_{\tau}\left(k_{lm},Z_{m}\right)}\right)^{\frac{1}{\psi}-\gamma}\left(F\left(\overline{K}-\hat{K}^{E\prime}_{\tau}\left(k_{lm},Z'\right)\right)-\hat{Q}_{\tau}\left(k_{lm},Z'\right)\left[k_{lm}-\hat{K}^{E\prime}_{\tau}\left(k_{lm},Z'\right)\right]\right)^{-\frac{1}{\psi}}\hat{Q}_{\tau}\left(k_{lm},Z'\right)\right]$$

$$v^{E}_{lm}=\left[\left(1-\beta\right)\left(Z_{m}F\left(k_{lm}\right)-q_{lm}\left[k_{lm}-K_{l}\right]\right)^{1-\frac{1}{\psi}}+\beta\hat{\zeta}^{E}_{\tau}\left(k_{lm},Z\right)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$
$$v^{H}_{lm}=\left[\left(1-\beta\right)\left(F\left(\overline{K}-k_{lm}\right)-q_{lm}\left[K_{l}-k_{lm}\right]\right)^{1-\frac{1}{\psi}}+\beta\hat{\zeta}^{H}_{\tau}\left(k_{lm},Z\right)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$
$$\text{where}\quad \hat{\zeta}^{i}_{\tau}\left(k_{lm},Z_{m}\right)=\mathbb{E}\left[\hat{V}^{i}_{\tau}\left(k_{lm},Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}},\ \text{for } i\in\{E,H\}$$

   (b) This is effectively just solving the first two equations for k_{lm} and q_{lm} and then backing out the implied v^{E}_{lm} and v^{H}_{lm} from the last two.
   (c) Set $\left\{\hat{M}^{K}_{\tau+1}\left(l,m\right),\hat{M}^{VE}_{\tau+1}\left(l,m\right),\hat{M}^{VH}_{\tau+1}\left(l,m\right),\hat{M}^{Q}_{\tau+1}\left(l,m\right)\right\}=\left\{k_{lm},v^{e}_{lm},v^{h}_{lm},q_{lm}\right\}$, with $\hat{M}^{j}_{\tau+1}\left(l,m\right)$ referring to the row-l, column-m entry of $\hat{M}^{j}_{\tau+1}$.
   (d) Check if $\max_{l,m,j}\left|\hat{M}^{j}_{\tau}\left(l,m\right)-\hat{M}^{j}_{\tau}\left(l,m\right)\right|\leq\varepsilon$, for some small ε. If yes, stop the iteration. If not, go to (a).

D.3 Capital + Bond

The general approach to solving the model with trade in capital and a risk-free bond is identical to that with trade in only capital. In terms of additional equilibrium objects, there are now also the quantities of bond holdings of both agents and the price of the risk-free bond to be determined in equilibrium. And the equilibrium conditions are augmented with two Euler equations (one for the bond holdings of each agent) and the market clearing condition for the bonds in zero net supply. Also, even though the main text shows that it is enough to track the entrepreneur's wealth level as the only endogenous aggregate state, this aggregate state in period t will not be "predetermined" in $t-1$ and will itself depend on the realized shocks in t. For numerical stability, and for a natural fit with the time-iteration method outlined above, as a first approach we choose $\mathrm{Y} = \left(K^E, B^E, Z\right)$ to track as the aggregate state.

That is, we can now think of the equilibrium as the functions $f \equiv \left\{\left\{C^i(\mathrm{Y}), K^{i\prime}(\mathrm{Y}), B^{i\prime}(\mathrm{Y}), V^i(\mathrm{Y})\right\}_{i\in\{E,H\}}, Q(\mathrm{Y}), q^f(\mathrm{Y})\right\}$ that satisfy the following system for any $\mathrm{Y} = (K, B, Z) \in \left[0, \overline{K}\right] \times [-\bar{B}, \bar{B}] \times \mathcal{Z}$, for some $\bar{B} > 0$:

$$\left[Q(\mathrm{Y}) - Z^i F'\left(K^{i\prime}(\mathrm{Y})\right)\right]\left(C^i(\mathrm{Y})\right)^{-\frac{1}{\psi}} =$$

$$\beta\mathbb{E}\left[\left(\frac{V^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)}{\zeta^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(C^{i\prime}\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)\right)^{-\frac{1}{\psi}} Q\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)\right], \text{ for } i \in \{E, H\}$$

$$q^f(\mathrm{Y})\left(C^i(\mathrm{Y})\right)^{-\frac{1}{\psi}} = \beta\mathbb{E}\left[\left(\frac{V^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)}{\zeta^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(C^{i\prime}\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)\right)^{-\frac{1}{\psi}}\right], \text{ for } i \in \{E, H\}$$

$$\text{where} \quad \zeta^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z\right) = \mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$C^i(\mathrm{Y}) = Z^i F\left(K^{i\prime}(\mathrm{Y})\right) - Q(\mathrm{Y})\left[K^{i\prime}(\mathrm{Y}) - K^i\right] - q^f(\mathrm{Y})\, B^{i\prime}(\mathrm{Y}) + B^i, \text{ for } i \in \{E, H\}$$

$$\text{with } Z^E = Z,\ Z^H = 1 \text{ and } K^H = \overline{K} - K^E, B^H = -B^E$$

$$V^i(\mathrm{Y}) = \left[(1-\beta)\left(C^i(\mathrm{Y})\right)^{1-\frac{1}{\psi}} + \beta\mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \text{ for } i \in \{E, H\}$$

$$K^{E\prime}(\mathrm{Y}) + K^{H\prime}(\mathrm{Y}) = \overline{K}$$
$$B^{E\prime}(\mathrm{Y}) + B^{H\prime}(\mathrm{Y}) = 0$$

This system can again be collapsed, now into a collection of four Euler equations (in capital and bonds for both agents) and the determination of the value functions to yield a system in $f = \left\{K^{E\prime}(\mathrm{Y}), B^{E\prime}(\mathrm{Y}), V^E(\mathrm{Y}), V^H(\mathrm{Y}), Q(\mathrm{Y}), q^f(\mathrm{Y})\right\}$. We again solve for approximations to these equilibrium objects, denoted as $\hat{f}$, by time iteration. A key difference is that since there is now an additional, continuous dimension in the aggregate state space, we discretize the space $[-\bar{B}, \bar{B}]$ into n_B points. And an equilibrium function f^j is now approximated by a collection of n_z piecewise (bi-)linear functions $\left\{\hat{f}^j_{Z_m}\right\}_{Z_m\in\mathcal{Z}}$ defined on a *two-dimensional* domain, with $\hat{f}^j_{Z_m}(K, B) \approx f^j(K, B, Z_m)$, and each $\hat{f}^j_{Z_m}$ being defined by a matrix of dimensions $n_K \times n_B$ – its values on the discretized points in the (K, B)-space, given $Z = Z_m$. Or, equivalently, the approximant $\hat{f}^j$ is defined by a $n_K \times n_B \times n_Z$ matrix $\hat{M}^j$of values with the (l, n, m)-entry corresponding to $\hat{f}^j_{Z_m}(K_l, B_n)$, and the computational task amounts to finding the $n_K \times n_B \times n_Z$ matrices such that the equilibrium conditions are satisfied by the implied approximants $\hat{f}$ at each point $(K_l, B_n, Z_m) \in \{K_l\}_{l\in\{1,\ldots,n_K\}} \times \{B_n\}_{n\in\{1,\ldots,n_B\}} \times \mathcal{Z}$. The time iteration algorithm follows precisely the same steps as before.

D.4 Complete Markets

The solution algorithm for the model with Complete Markets is a natural extension of the "Capital Only" model, along the same lines as the "Capital+Bond" model. In terms of additional equilibrium objects, instead of the two agents' bond holdings and the bond price, there are now the acquired holdings of the n_Z Arrow-Debreu securities of both agents and the n_Z prices of each of these securities to be determined at every state in equilibrium. And the equilibrium conditions are augmented by $2n_Z$ optimality conditions (one for the security holdings of each agent) and the market clearing condition for each of the n_Z securities in zero net supply. Again, even though it would be enough to track the entrepreneur's wealth level as the only endogenous aggregate state, we choose $\mathrm{Y} = \left(K^E, B^E, Z\right)$ to track as the aggregate state. For any realization of Z_m, B^E now refers to the amount of the Arrow-Debreu security which pays if $Z_m = Z$ is realized that the entrepreneur acquired in the previous period.

That is, we can now think of the equilibrium as the functions $f \equiv \left\{\left\{C^i(\mathrm{Y}), K^{i\prime}(\mathrm{Y}), \left\{B_s^{i\prime}(\mathrm{Y})\right\}_{s\in\mathcal{Z}}, V^i(\mathrm{Y})\right\}_{i\in\{E,H\}}, Q(\mathrm{Y}), \{q_s(\mathrm{Y})\}_{s\in\mathcal{Z}}\right\}$, with s denoting the realization of next period's exogenous state (here, just the value of Z'), and $B_{Z_m}^{i\prime}(\mathrm{Y})$ referring to the amount of Arrow-Debreu security paying if $Z' = Z_m$ acquired by agent i whenever the current state is Y. And these functions must satisfy the following system for any $\mathrm{Y} = (K, B, Z) \in \left[0, \overline{K}\right] \times [-\bar{B}, \bar{B}] \times \mathcal{Z}$, for some $\bar{B} > 0$:

$$\left[Q(\mathrm{Y}) - Z^i F'\left(K^{i\prime}(\mathrm{Y})\right)\right]\left(C^i(\mathrm{Y})\right)^{-\frac{1}{\psi}} =$$

$$\beta\mathbb{E}\left[\left(\frac{V^i\left(K^{E\prime}(\mathrm{Y}), B_{Z'}^{E\prime}(\mathrm{Y}), Z'\right)}{\zeta^i\left(K^{E\prime}(\mathrm{Y}), \{B_s^{E\prime}(\mathrm{Y})\}_{s\in\mathcal{Z}}, Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(C^{i\prime}\left(K^{E\prime}(\mathrm{Y}), B_{Z'}^{E\prime}(\mathrm{Y}), Z'\right)\right)^{-\frac{1}{\psi}} Q\left(K^{E\prime}(\mathrm{Y}), B_{Z'}^{E\prime}(\mathrm{Y}), Z'\right)\right], \text{ for } i \in \{E, H\}$$

$$\frac{q_{Z_m}(\mathrm{Y})}{\pi(Z_m|Z)}\left(C^i(\mathrm{Y})\right)^{-\frac{1}{\psi}}$$

$$= \beta\left(\frac{V^i\left(K^{E\prime}(\mathrm{Y}), B_{Z_m}^{E\prime}(\mathrm{Y}), Z_m\right)}{\zeta^i\left(K^{E\prime}(\mathrm{Y}), \{B_s^{E\prime}(\mathrm{Y})\}_{s\in\mathcal{Z}}, Z\right)}\right)^{\frac{1}{\psi}-\gamma}\left(C^{i\prime}\left(K^{E\prime}(\mathrm{Y}), B_{Z_m}^{E\prime}(\mathrm{Y}), Z_m\right)\right)^{-\frac{1}{\psi}}, \text{ for } i \in \{E, H\}, Z_m \in \mathcal{Z}$$

$$\text{where} \quad \zeta^i\left(K^{E\prime}(\mathrm{Y}), \left\{B_s^{E\prime}(\mathrm{Y})\right\}_{s\in\mathcal{Z}}, Z\right) = \mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), B_{Z'}^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$C^i(\mathrm{Y}) = Z^i F\left(K^{i\prime}(\mathrm{Y})\right) - Q(\mathrm{Y})\left[K^{i\prime}(\mathrm{Y}) - K^i\right] - \sum_{s\in\mathcal{Z}} q_s(\mathrm{Y}) B_s^{i\prime}(\mathrm{Y}) + B^i, \text{ for } i \in \{E, H\}$$

$$\text{with } Z^E = Z, Z^H = 1 \text{ and } K^H = \overline{K} - K^E, B^H = -B^E$$

$$V^i(\mathrm{Y}) = \left[(1-\beta)\left(C^i(\mathrm{Y})\right)^{1-\frac{1}{\psi}} + \beta\mathbb{E}\left[V^i\left(K^{E\prime}(\mathrm{Y}), B_{Z'}^{E\prime}(\mathrm{Y}), Z'\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \text{ for } i \in \{E, H\}$$

$$K^{E\prime}(\mathrm{Y}) + K^{H\prime}(\mathrm{Y}) = \overline{K}$$

$$B_{Z_m}^{E\prime}(\mathrm{Y}) + B_{Z_m}^{H\prime}(\mathrm{Y}) = 0, \text{ for } Z_m \in \mathcal{Z}$$

This system can again be collapsed, now into a collection of $2 + 2n_Z$ optimality conditions (in capital and bonds for both agents) and the determination of the value functions to yield a system in $f = \left\{K^{E\prime}(\mathrm{Y}), \{B_s^{E\prime}(\mathrm{Y})\}_{s\in\mathcal{Z}}, V^E(\mathrm{Y}), V^H(\mathrm{Y}), Q(\mathrm{Y}), \{q_s(\mathrm{Y})\}_{s\in\mathcal{Z}}\right\}$. With a slight abuse of notation, instead of solving for $q_{Z_m}(\mathrm{Y})$ it is better to solve for the probability adjusted security prices $\frac{q_{Z_m}(\mathrm{Y})}{\pi(Z_m|Z)}$ because small probabilities $\pi(Z_m|Z)$ can potentially lead to very small prices, close to machine epsilon and cause imprecision in computation. Further, as shown in the main text, because of market completeness, production efficiency holds at all times: $K^{E\prime}\left(K^E, B^E, Z\right) = K^*(Z)$, where $K^*(Z)$ solves $ZF'(K^*) = F'\left(\overline{K} - K^*\right)$. So $K^{E\prime}(\mathrm{Y})$ can be determined separately, leaving a system of $3 + 2n_Z$ equations in the remaining equilibrium values. We again solve for approximations to all these $4 + 2n_Z$ equilibrium objects, denoted as $\hat{f}$, by time iteration, with each equilibrium function f^j approximated by a collection of n_z piecewise (bi-)linear functions $\left\{\hat{f}_{Z_m}^j\right\}_{Z_m\in\mathcal{Z}}$ defined on a *two-dimensional* domain. The collection of approximations to $\left\{B_s^{E\prime}(\mathrm{Y})\right\}_{s\in\mathcal{Z}}$ (analogously for $\{q_s(\mathrm{Y})\}_{s\in\mathcal{Z}}$) is now $\left\{\hat{f}_{Z_m,Z_p}^B\right\}_{Z_m,Z_p\in\mathcal{Z}}$, each $\hat{f}_{Z_m,Z_p}^B$ being a bilinear function on the (K, B)-domain, with $\hat{f}_{Z_m,Z_p}^B(K, B) \approx B_{Z_p}^{E\prime}(K, B, Z_m)$. And this collection is defined by a $n_K \times n_B \times n_Z \times n_Z$ matrix $\hat{M}^B$of values with the (l, n, m, p)-entry corresponding to $\hat{f}_{Z_m,Z_p}^B(K_l, B_n)$. And again the computational task amounts to finding the 4 $n_K \times n_B \times n_Z$ matrices $\left\{\hat{M}^K, \hat{M}^{VE}, \hat{M}^{VH}, \hat{M}^Q\right\}$ and 2 $n_K \times n_B \times n_Z \times n_Z$ matrices $\left\{\hat{M}^B, \hat{M}^q\right\}$ such that the equilibrium conditions are satisfied by the implied approximants $\hat{f}$ at each point $(K_l, B_n, Z_m) \in \{K_l\}_{l\in\{1,\ldots,n_K\}} \times \{B_n\}_{n\in\{1,\ldots,n_B\}} \times \mathcal{Z}$. The time iteration algorithm follows precisely the same steps as before.

D.5 Financial Market Shocks

The solution algorithm for the model with Complete Markets is the natural combination of the approaches to the "Capital+Bond" and "Complete Markets" models. The aggregate state of the economy is now $\mathrm{Y} = \left(K^E, B^E, (Z,\phi)\right)$ where we think of the exogenous state (Z,ϕ) as a one-dimensional object, with $2n_Z$ possible realizations. Again "collapsing" the equilibrium conditions at any Y, the equilibrium objects are now $\left\{K^{E\prime}(\mathrm{Y}), V^E(\mathrm{Y}), V^H(\mathrm{Y}), Q(\mathrm{Y})\right\}$ and the entrepreneur's risk-free bond holdings and the risk-free bond price $\left\{B^E(\mathrm{Y}), q^f(\mathrm{Y})\right\}$ whenever $\phi = 1$ and the $2n_Z$ entrepreneur's Arrow-Debreu security holdings and their $2n_Z$ (probability adjusted) prices $\left\{B^E_s(\mathrm{Y}), q^s_s(\mathrm{Y})\right\}_{s\in\mathcal{Z}\times\{0,1\}}$ whenever $\phi = 0$. At any Y with $\phi = 0$, the effective equilibrium conditions are those stated for the "Complete Markets" model above, and otherwise those for the "Capital+Bond" model (of course, taking into account the future possibility of both $\phi = 0$ or $\phi = 1$).

Again, we solve for approximations to all these equilibrium objects, denoted as $\hat{f}$, by time iteration, with each equilibrium function f^j approximated by a collection of n_z piecewise (bi-)linear functions $\left\{\hat{f}^j_{Z_m}\right\}_{Z_m\in\mathcal{Z}}$ defined on a *two-dimensional* domain. It is convenient to combine the quantities of risk-free bonds and Arrow-Debreu securities acquired by the entrepreneur (similarly, their prices) at any point on the discretized state space $\{K_l\}_{l\in\{1,\dots,n_K\}} \times \{B_n\}_{n\in\{1,\dots,n_B\}} \times \mathcal{Z}_{\prec}$, with $\mathcal{Z}_\phi \equiv \mathcal{Z}\times\{0,1\}$, into a $n_K \times n_B \times 2n_Z \times 2n_Z$ matrix where the (l,n,m,p)-entry refers to $B^{E\prime}_{(Z,\phi)_p}\left(K_l, B^E_n, (Z,\phi)_m\right)$, with $(Z,\phi)_m = (Z,1)$ whenever $1 \le m \le n_Z$ and $(Z,\phi)_m = (Z,0)$ if $n_Z < m \le 2n_Z$. And we will use the convention that $B^{E\prime}_{(Z,\phi)_p}\left(K_l, B^E_n, (Z,\phi)_m\right) = B^{E\prime}_{(Z,\phi)_r}\left(K_l, B^E_n, (Z,\phi)_m\right), \forall p, r \in \{1,\dots,2n_Z\}$ whenever $1 \le m \le n_Z$ – of course, standing in for the fact that the securities carried forward into each of the following states must be equal across future states whenever $\phi = 1$.

And again the computational task amounts to finding the four $n_K \times n_B \times 2n_Z$ matrices $\left\{\hat{M}^K, \hat{M}^{VE}, \hat{M}^{VH}, \hat{M}^Q\right\}$ and two $n_K \times n_B \times 2n_Z \times 2n_Z$ matrices $\left\{\hat{M}^B, \hat{M}^q\right\}$ such that the equilibrium conditions are satisfied by the implied approximants $\hat{f}$ at each point $(K_l, B_n, (Z,\phi)_m) \in \{K_l\}_{l\in\{1,\dots,n_K\}} \times \{B_n\}_{n\in\{1,\dots,n_B\}} \times \mathcal{Z}_\phi$. The general time iteration algorithm follows precisely the same steps as before.